DCAS-TDR-62-139

$N-63-4-4$

17505

Electromagnetic Wave Propagation Through Magnetoactive Plasmas

1 FEBRUARY 1962

Prepared by R. MASON and R. R. GOLD

Prepared for DEPUTY COMMANDER AEROSPACE SYSTEMS
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Inglewood, California

LABORATORIES DIVISION • AEROSPACE CORPORATION

DCAS-TDR-62-139

ELECTROMAGNETIC WAVE PROPAGATION
THROUGH MAGNETOACTIVE PLASMAS

NA

Prepared by

R. Mason and R. R. Gold

AEROSPACE CORPORATION
El Segundo, California

1 February 1962

Prepared for

DEPUTY COMMANDER AEROSPACE SYSTEMS
AIR FORCE SYSTEMS COMMAND
UNITED STATES AIR FORCE
Inglewood, California

ABSTRACT

The general problem of the propagation of a plane harmonic electromagnetic wave through an ionized gas is considered, including collisions, some form of electron density distribution, and a magnetostatically induced anisotropy. The present state of the art is examined first, including an outline of the WKB approximation and a summary of electron density distributions for which complete wave theory solutions exist. Then, specific analytical solutions and appropriate numerical calculations are derived to evaluate the effect of the applied magnetic field on propagation. A detailed parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

CONTENTS

SECTION I

INTRODUCTION

The problem of electromagnetic wave propagation in an anisotropic medium has been considered in several diverse fields of study. Initial interest centered about investigations in crystal optics (see Born and Wolf[1]). More recently, the introduction of anisotropic ferrite media in microwave device applications resulted in further treatment of the subject. Plasma physicists have employed microwave beams to produce diffuse plasmas and for diagnostic purposes. They have contributed extensively to the microscopic theory of propagation in magnetoactive[*] plasmas in such related fields as plasma diagnostics, astrophysics, and controlled nuclear reactions. Considerable work has been done in the field of ionic radio-wave propagation, the so-called magneto-ionic theory. The primary interest there is the study of reflection of electromagnetic waves from the inhomogeneous ionosphere in the presence of a constant magnetic field. Of interest also is the use of radio waves as a probing tool to examine the physics of the ionosphere itself. With the advent of space exploration, the problem of communication with vehicles through the plasma sheath or wake has arisen, and the application of magnetic fields has been suggested as a possible solution.

The present study is formulated in terms of ionized gas parameters, using a macroscopic approach in which the medium is characterized by simple constitutive relations. Although the emphasis, therefore, is in the field of plasma physics, many of the general considerations and methods of solution may be appropriate to the research work involving solids mentioned above or, for example, to the study of acoustical wave propagation in inhomogeneous fluids.

---

[*] The use of this term is convenient to indicate the presence of an applied magnetic field.

The basic equations governing the propagation of electromagnetic waves in inhomogeneous, nonuniform, magnetoactive plasmas are derived in the next section. In Section III, a homogeneous plasma is considered and the essential elements of ray theory are reviewed including the limiting condition as prescribed by the WKB approximation. The general inhomogeneous plasma problem is examined in Section IV, and a summary of electron density distributions for which there are full wave solutions is presented. Appropriate analytical solutions and numerical calculations are obtained in Sections V, VI, and VII, which provide the basis for the present preliminary evaluation of the effect of an applied magnetic field on propagation. Detailed quantitative (parametric) results are obtained in Section V, assuming a homogeneous plasma slab and normal incidence, normal applied uniform magnetic field. Specific exemplary flight conditions are examined also on this basis as is the initial consideration of a proposed experimental shock-tube program. The more realistic inhomogeneous, nonuniform plasma problem is analyzed in Sections VI and VII.

SECTION II

BASIC EQUATIONS

Maxwell's point formulation of the electromagnetic field equations is assumed to be valid for an inhomogeneous magnetoactive medium (Stratton[2]):

$$\nabla \times \vec{E}' = -\frac{\partial \vec{B}'}{\partial t} \qquad , \qquad \nabla \times \vec{H}' = \frac{\partial \vec{D}'}{\partial t} + \vec{J}' \qquad , \qquad (1, 2)$$

$$\nabla \cdot \vec{D}' = \rho \qquad , \qquad \nabla \cdot \vec{B}' = 0 \qquad , \qquad (3, 4)$$

where $\vec{E}'$ and $\vec{H}'$ are the electric and magnetic field intensities, respectively, $\vec{D}'$ is the electric displacement, $\vec{B}'$ is the magnetic induction, and ρ and $\vec{J}'$ are the charge and current densities, respectively (rationalized mks units are used). The continuity of charge condition relates ρ and $\vec{J}'$:

$$\nabla \cdot \vec{J}' + \frac{\partial \rho}{\partial t} = 0 \qquad . \qquad (5)$$

Two of these equations are not independent, however, as can be shown in the following way:

$$\nabla \cdot (\nabla \times \vec{E}') = 0 = -\frac{\partial}{\partial t} (\nabla \cdot \vec{B}') \qquad ,$$

$$\nabla \cdot (\nabla \times \vec{H}') = 0 = \frac{\partial}{\partial t} (\nabla \cdot \vec{D}') + \nabla \cdot \vec{J}' = \frac{\partial}{\partial t} (\nabla \cdot \vec{D}' - \rho) \qquad .$$

Hence, we may regard (1), (2), and (5) as the basic independent equations and consider (3) and (4) as initial conditions that are valid at all times, if valid initially. The resultant system of seven independent scalar equations

-3-

in 16 scalar unknowns is made determinate by introducing the macroscopic properties of the medium which empirically relate $\vec{E}'$, $\vec{D}'$, and $\vec{J}'$ to $\vec{H}'$ and $\vec{E}'$:

$$\vec{B}' = \mu\vec{H}' \quad , \quad \vec{D}' = \epsilon\vec{E}' \quad , \quad \vec{J}' = \sigma\vec{E}' \quad . \quad (6,7,8)$$

The inductive capacities μ and ϵ and the electrical conductivity σ are, in the most general cases, time- and space-dependent tensors that characterize the electromagnetic properties of the medium.

For a slightly ionized gas, the inductive capacities are, for all practical purposes, the same as the free space values μ_0 and ϵ_0. If we neglect the motion of the more massive ions under the influence of microwave irradiation, then $\vec{J}'$ is given by [rewriting (6) and (7) for completeness]:

$$\vec{J}' = Ne\vec{v}' \quad , \quad \vec{B}' = \mu_0\vec{H}' \quad , \quad \vec{D}' = \epsilon_0\vec{E}' \quad , \quad (9,10,11)$$

where $\vec{v}'$ is the electron drift velocity, N is the number density of electrons, and e is the charge. We will assume harmonic time dependence of the form $\exp(i\omega t)$ for all time-varying vectors. Thus, for any such quantity $\vec{V}'$:

$$\vec{V}'(x, y, z, t) = \vec{V}(x, y, z)e^{i\omega t} \quad , \quad (12)$$

where ω is the angular wave frequency. It will be convenient to introduce a modified magnetic field intensity:

$$\vec{H}\Big| = \left(\frac{\mu_0}{\epsilon_0}\right)^{1/2} \vec{H} \quad , \quad (13)$$

where $(\mu_0/\epsilon_0)^{1/2} = c_0\mu_0$ is the free space impedance and c_0 is the velocity of light in vacuo.

-4-

The electrical conductivity and, therefore, the relationship between $\vec{J}$ and $\vec{E}$ for the plasma, is determined from the equation of motion for an average electron in some region of space in which is established, in general, a steady, spatially dependent biasing magnetic field of induction $\vec{B}_0$:

$$m\frac{d\vec{v'}}{dt} + m\omega_c \vec{v'} = e(\vec{E'} + \vec{v'} \times \vec{B}_0) \quad , \tag{14}$$

where m is the electron mass, ω_c is the average electron collision frequency, and the force exerted by the magnetic field of the wave is neglected relative to the force $\vec{E'}e$. Assuming harmonic time dependence, we see, using (9), that:

$$\vec{J} = \frac{i\epsilon_0 \omega X}{U(U^2 - Y^2)}\left[-U^2\vec{E} + Y^2(\vec{E} \cdot \vec{i}_B)\vec{i}_B + iYU(\vec{E} \times \vec{i}_B) \right] \quad , \tag{15}$$

where $\vec{i}_B$ is the unit vector $(\vec{B}_0 = B_0\vec{i}_B)$ whose direction cosines relative to the Cartesian coordinate system are ℓ, m, and n, the angular plasma frequency ω_p, and the angular cyclotron frequency at which the electrons gyrate due to the externally applied magnetic field $\vec{\omega}_B$ are given by:

$$\omega_p^2 = \frac{Ne^2}{\epsilon_0 m} \quad , \qquad \vec{\omega}_B = \frac{e\vec{B}_0}{m} \quad , \tag{16a}$$

$$X = \left(\frac{\omega_p}{\omega}\right)^2 \quad , \qquad \vec{Y} = \frac{\vec{\omega}_B}{\omega} \quad , \qquad Z = \frac{\omega_c}{\omega} \quad , \qquad U = 1 - iZ \quad , \tag{16b}$$

and the vector $\vec{Y}$ acts in the opposite direction to $\vec{B}_0$ (since e < 0) where $Y = |\vec{Y}| = (|e|B_0/m\omega)$. Expressing (15) in matrix notation $\vec{J} = \hat{\sigma}\vec{E}$ exhibits the general character of the electrical conductivity for a magnetoactive

medium. Substituting Eqs. (10, 11, 13, 15, 16) into Maxwell's Eqs. (1) through (4), we obtain the following basic system:

$$\nabla \times \vec{E} = -in_0 \vec{H} \quad , \quad \nabla \times \vec{H} = in_0 \hat{K}\vec{E} \quad , \tag{17, 18}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad , \quad \nabla \cdot \vec{H} = 0 \quad , \tag{19, 20}$$

where the matrix K has components $K_{ij}(i, j = x, y, z)$,

$$\hat{K} = \begin{pmatrix} 1 - A(U^2 - \ell^2 Y^2) & A(-inYU + \ell mY^2) & A(imYU + \ell nY^2) \\ A(inYU + \ell mY^2) & 1 - A(U^2 - m^2 Y^2) & A(-i\ell YU + mnY^2) \\ A(-imYU + \ell nY^2) & A(i\ell YU + mnY^2) & 1 - A(U^2 - n^2 Y^2) \end{pmatrix} \quad , \tag{21}$$

$A = X/U(U^2 - Y^2)$, $n_0 = \omega/c_0 = 2\pi/\lambda_0$ is the free-space wave propagation constant, λ_0 the free-space wavelength, and $\vec{H}$ is given by Eq. (13). The complete system of differential equations for the wave motion is expressed by the two vector equations (17) and (18) in $\vec{E}$ and $\vec{H}$ where $\hat{K}$ is given by (21). It is noteworthy that the electromagnetic properties of the medium are completely characterized by (21). The orientation of $\vec{B}_0$ is arbitrary, and, since X, U, and Y may be space dependent, Eq. (21) will apply for the general case of an inhomogeneous, nonuniform, magnetoactive plasma. Additional discussion is given, in detail, by Ginzburg[3] and by Budden.[4]

SECTION III

PROPAGATION IN HOMOGENEOUS, UNIFORM, MAGNETOACTIVE PLASMAS

Eliminating $\vec{H}$ from (17) and (18), we obtain the following wave-like equation for $\vec{E}$:

$$\nabla \times \nabla \times \vec{E} = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} = n_0^2 \hat{K} \vec{E} \quad , \tag{22}$$

where the matrix $\hat{K}$ is given in (21). Consider now the solutions to this equation which represent possible plane waves in an infinite homogeneous, uniform, magnetoactive plasma. Since $\hat{K}$ is constant, we may assume characteristic solutions of the form:

$$\vec{E} = \vec{E}^{(1)} e^{-in_0 \vec{n} \cdot \vec{r}} \quad , \tag{23}$$

where $\vec{n}$ is the so-called wave normal (i.e., normal to the plane wave front) which identifies the direction of propagation, $\vec{E}^{(1)}$ is the constant amplitude, and $\vec{r}$ is the radius vector. In the general case, for a real frequency ω, the wave normal is a complex vector of the form $\vec{n} = \vec{p} - i\vec{q}$, where $\vec{p}$ and $\vec{q}$ are real. We will restrict our attention to the case of homogeneous plane waves, however, so that the planes of equal phase and amplitude coincide and the wave normal may be written in the simplified form:

$$\vec{n} = (\alpha - i\beta)\vec{t} \quad , \tag{24}$$

where $\vec{t}$ is a unit vector (real) in the propagation direction.

-7-

Substituting (23) and (24) into (22), we obtain:

$$(\alpha - i\beta)^2 \vec{t} \times (\vec{t} \times \vec{E}^{(1)}) = -\hat{K}\vec{E}^{(1)} \quad , \qquad (25)$$

which represents a system of three homogeneous algebraic equations for the three components of $\vec{E}^{(1)}$. The existence of a nontrivial solution of this system requires that the determinant of the coefficients vanish, from which one may obtain the required expression for the quantity $(\alpha - i\beta)^2$. It would appear on first examination of (25) that the resultant equation should be a cubic in $(\alpha - i\beta)^2$. If, however, we take the dot product of (25) with $\vec{t}$, the left-hand side will vanish (thus, the projection of the vector $\hat{K}\vec{E}^{(1)}$ in the propagation direction is zero). Since $\hat{K}$ is not a function of $(\alpha - i\beta)$, it follows that there exists a linear relation between the three components of $\vec{E}^{(1)}$ in the wave which must be independent of $(\alpha - i\beta)^2$. Without going through the details of writing out the full expression, we see, therefore, that the aforementioned condition for the existence of a nontrivial solution of the system (25), or, equivalently, the existence of solutions to (22) of the form (23), leads to a quadratic relation for the quantity $(\alpha - i\beta)^2$.

It is clear from the preceding remarks that the simplest way of deriving the desired expression for $n*$,

$$n* = \alpha - i\beta \quad , \qquad (26)$$

is to choose a coordinate system so that the positive z-axis is in the direction of $\vec{t}$. The z-component equation of (25) will not involve $n*$, and a quadratic relation for $n*^2$ will follow with a minimum of algebraic manipulation. There is no loss of generality using this effort-saving simplification, because the properties of the medium should not depend on the choice of coordinate system. Further, for an arbitrary orientation of the magnetic field, the y-axis can be selected perpendicular to $\vec{B}_0$. Therefore, we will assume that $\vec{B}_0$ is in the

-8-

x-z plane at an angle γ to the positive z-axis. The quadratic equation defining $n*^2$ will now be derived from (25) on the basis of the preceding remarks.

The three scalar equations (25) for the components of $\vec{E}^{(1)}$ reduce to:

$$\left(-n*^2 + K_{xx}\right)E_x^{(1)} + K_{xy}E_y^{(1)} + K_{xz}E_z^{(1)} = 0 \qquad , \tag{27a}$$

$$K_{yx}E_x^{(1)} + \left(-n*^2 + K_{yy}\right)E_y^{(1)} + K_{yz}E_z^{(1)} = 0 \qquad , \tag{27b}$$

$$K_{zx}E_x^{(1)} + K_{zy}E_y^{(1)} + K_{zz}E_z^{(1)} = 0 \qquad . \tag{27c}$$

The vanishing of the determinant of the coefficients of $\vec{E}^{(1)}$ leads to the following quadratic equation in $n*^2$:

$$[1 - A(U^2 - Y^2) - AY^2 \sin^2 \gamma] n*^4$$
$$+ \left\{-2(1 - AU^2)[1 - A(U^2 - Y^2)] + AY^2 \sin^2 \gamma\right\} n*^2$$
$$+ [(1 - AU^2)^2 - A^2 U^2 Y^2][1 - A(U^2 - Y^2)] = 0 \qquad . \tag{28}$$

Rearranging terms, substituting $A = X/U \ (U^2 - Y^2)$, and multiplying by $U \ (U^2 - Y^2)$, we obtain:

$$(U - X)[U(n*^2 - 1) + X]^2 + [-Y^2(U - X) - XY^2 \sin^2 \gamma](n*^2 - 1)^2$$
$$- XY^2 \sin^2 \gamma(n*^2 - 1) = 0 \qquad .$$

If we now make the substitution:

$$-Y^2(U - X) = -Y^2(U - X) \sin^2 \gamma - Y^2(U - X) \cos^2 \gamma$$

in the second term and divide by $(U - X)(n*^2 - 1)^2$, we obtain a quadratic equation for the quantity $U + X/(n*^2 - 1)$, from which it follows that:

$$n_{\pm}^{*2} = 1 - \frac{X}{U - \left[Y_T^2/2(U - X) \right] \mp \left\{ \left[Y_T^4/4(U - X) \right] + Y_L^2 \right\}^{1/2}} \quad , \quad (29)$$

where Y_T is the transverse component of the vector $\vec{Y}$, and Y_L is the longitudinal component; i.e.,

$$Y_L = Y \cos \gamma \quad , \quad Y_T = Y \sin \gamma \quad . \quad (30)$$

Equation (29) is the classical Appleton-Hartree formula for the refractive index of a homogeneous, uniform, magnetoactive medium, $n* = \alpha - i\beta$. Each value of $n_{\pm}^{*2}$ represents a pair of waves traveling both in the positive and the negative z-directions. It is customary to take $n_{\pm}^* = + \left(n_{\pm}^{*2} \right)^{1/2}$ for both waves, as determined by the positive and negative signs in (29), in which case the general solution of (22) becomes:

$$\vec{E} = \vec{E}^{(1)} e^{-in_0 n*z} + \vec{E}^{(2)} e^{in_0 n*z} \quad , \quad (31)$$

where the first term represents a wave traveling in the positive z-direction and the second term represents a wave traveling in the negative z-direction (the real numbers α and β are taken to be positive). The physical meaning of the two possible waves in the medium corresponding to $n_{\pm}^*$ is described best in terms of the polarization of the waves, i.e., the relationship between the components E_x and E_y with time. In general, both waves are elliptically polarized. It will be shown subsequently that, when $Y_T = 0$, the plus sign (usually referred to as the "ordinary" wave) represents a circularly polarized wave whose electric field vector is rotating clockwise with time, looking along

-10-

the field, i.e., in the positive z-direction, while the minus sign ("extra-ordinary" wave) denotes a counterclockwise rotation.

The electromagnetic properties of the medium are completely charac-terized by the complex index of refraction. Specific calculations have been made for a wide range of values of the three frequency ratios (assuming $Y_T = 0$) in the present study. Rather than presenting these extensive numer-ical results, we will discuss the general propagation properties of the plasma quantitatively in Section V by means of the more meaningful consideration of a particular boundary value problem associated with a homogeneous plasma slab. A very thorough discussion of the general properties of the Appleton-Hartree equation (29) is given by Ratcliffe,[5] Booker,[6] Budden,[4] and Ginzburg.[3] The reader's attention is also directed to the work of Shkarofsky[7] in which a variable electron collision frequency is considered.

Equation (29) was derived above for our discussion of plane wave propagation in an infinite, homogeneous, magnetoactive medium where the direction of propagation is arbitrary. It was convenient to choose the posi-tive z-axis in the direction of propagation for this purpose. Subsequently, however, we will be concerned with the propagation of plane waves from one medium to another across an abrupt plane boundary. It is clearly no longer convenient to use this approach, therefore, since the direction of propagation and, hence, the coordinate system would be different in each medium. Apply-ing a simple rotation of the axes, we arrive at the fixed coordinate system shown in Fig. 1. A plane wave is obliquely incident on the free space-plasma interface $z = 0$ with its wave normal in the x-z plane at an angle θ_I to the positive z-axis, $\vec{i}_B$ is at an angle δ from the positive z-axis, and the wave normal in the medium ($z > 0$) is at an angle θ. Equation (29) for $n*^2$ remains unchanged; however, the components of $\vec{Y}$ given in (30) are expressed now in terms of the angle $\gamma = \delta - \theta$.

-11-

Snell's law, which results from the requirement that the tangential components of $\vec{E}$ be continuous at $z = 0$, states that:

$$\sin \theta_I = n^* \sin \theta \quad , \tag{32}$$

where n^* and θ may refer to either transmitted wave. The solutions in the plasma region given by (23) are now of the form:

$$\vec{E} = \vec{E}^{(1)} \exp[-in_0(\alpha - i\beta)(x \sin \theta + z \cos \theta)] = \vec{E}^{(1)} \exp[-in_0(x \sin \theta_I + qz)] \quad , \tag{33}$$

$$q = n^* \cos \theta \quad . \tag{34}$$

The initial problem of determining n^* and θ by solving (29) and (32) simultaneously is greatly simplified using the relation (34) introduced by Booker.[8] Substituting (33) into (22) and proceeding as before, we obtain Booker's quartic for q:

$$
\left.
\begin{aligned}
F(q) &= c_1 q^4 + c_2 q^3 + c_3 q^2 + c_4 q + c_5 = 0 \quad , \\[2mm]
c_1 &= U(U^2 - Y^2) + X(n^2 Y^2 - U^2) \quad , \qquad c_2 = 2 \ln SXY^2 \quad , \\[2mm]
c_3 &= -2U(U - X)(C^2 U - X) + 2Y^2(C^2 U - X) + XY^2(1 - C^2 n^2 + S^2 l^2) \quad , \\[2mm]
c_4 &= -2C^2 \ln SXY^2 \quad , \\[2mm]
c_5 &= (U - X)(C^2 U - X)^2 - C^2 Y^2(C^2 U - X) - l^2 S^2 C^2 XY^2 \quad ,
\end{aligned}
\right\} \tag{35}
$$

-12-

where $S = \sin \theta_I$, $C = \cos \theta_I$, and use is made of the fact that the component of $\vec{Y}$ in the direction of the wave normal is $Y_L = Y (1S + qn)(q^2 + S^2)^{-1/2}$. The quartic has, in general, four distinct roots corresponding to a pair of waves traveling in the positive z-direction and a pair traveling in the direction of negative z, the former being of present interest. The two values of q for these waves then can be used to determine the corresponding values of n^* and θ from the following relations, obtained from Eqs. (32) and (34):

$$ n^{*2} = q^2 + S^2 \qquad , \qquad \tan \theta = \frac{S}{q} \qquad . \tag{36} $$

There are three cases for which (35) reduces to a quadratic equation in q^2. If $\theta_I = 0$ (normal incidence), the coefficients c_2 and c_4 vanish and the solutions for $q = n^*$ are those given by the Appleton-Hartree equation (29). The remaining two cases are for $\vec{B}_0$ assumed to lie in the y-z and x-y planes, respectively, (see Budden[4]) and are not of current interest.

Whenever n^* is varying locally, a reflection process is taking place so that the forward- and backward-traveling waves are in general coupled. This process is very weak in a "slowly changing" medium, where the electron density does not vary significantly with position, in which case the coupling may be neglected, except in the neighborhood of certain reflection points at which coupling is particularly strong. Such reflection points are found by setting $dq/dz = 0$ (Mitra[9]). This brief description is the basis for the so-called WKB approximate solutions for the electromagnetic fields at nearly all points in a "slowly changing" medium. Both the mathematical and physical consequences of this procedure will be clarified considerably in the next section when a specific problem is formulated and discussed. It is apparent, however, that propagation in homogeneous plasmas is governed by the well-known laws of geometrical optics, i.e., ray theory. As we have indicated, in general terms, approximate solutions can be obtained at nearly all points

in a slowly varying medium by assuming that the laws of geometrical optics apply at each point. In this sense the WKB solution is often referred to as a mathematical expression of ray theory. Several additional references of interest are Gershman, Ginzburg, and Denisov,[10] Pitteway,[11] Mitra,[12] Pisareva,[13] and Haselgrove.[14,15]

SECTION IV

PROPAGATION IN INHOMOGENEOUS, UNIFORM,
MAGNETOACTIVE PLASMAS

Although our primary purpose is to review the problem of electro-magnetic wave propagation in an inhomogeneous anisotropic (in particular a magnetoactive) plasma, it will be useful to consider also the corresponding isotropic case. For an isotropic medium $K_{ij} = K\delta_{ij}$, where K is now a scalar point function and δ_{ij} is the Kronecker delta, the following two equations can readily be derived from (17) and (18):

$$\nabla^2\vec{E} - \nabla(\nabla \cdot \vec{E}) + n_0^2 K(\omega, \vec{r})\vec{E} = 0 \quad , \tag{37}$$

$$\nabla^2 \vec{H} + \frac{1}{K(\omega, \vec{r})} [\nabla K(\omega, \vec{r}) \times (\nabla \times \vec{H})] + n_0^2 K(\omega, \vec{r})\vec{H} = 0 \quad , \tag{38}$$

where (20) is used in obtaining the latter result. Only one of these equations need be solved, because when either $\vec{E}$ or $\vec{H}$ is known, the remaining field intensity can be found from the appropriate field equation, (17) or (18). Although (37) is the equation most often used for this purpose, at times (38) is more convenient, depending on the nature of the problem, as will be noted later in this discussion. The general analysis of propagation in inhomogeneous media can proceed in a number of directions, particularly in view of the variety of assumptions for the function $K(\omega, \vec{r})$ which are possible. We will restrict our attention to plane-layered or stratified media, in which case $K = K(\omega, z)$. The propagation of waves in media whose properties are constant on spherical (or cylindrical) surfaces is similar in many respects to that in a plane-layered medium as observed by Ginzburg[3] (Sections 34 and 36).

Consider now the propagation of a plane wave across an abrupt, plane free space-plasma or plasma-plasma boundary, $z = 0$, with its wave normal in the x-z plane (at an angle θ_I to the positive z-axis in the region $z < 0$). The applied magnetic field vector (Fig. 1) will, in general, lie in the x-z plane and is assumed constant. Since the plasma is stratified (in the region $z > 0$) so that its properties are functions only of z, all field quantities will contain the factor $\exp(-in_0 x \sin \theta_I)$ by virtue of Snell's law. We will omit this factor in the present discussion in the same way that the time factor $\exp(i\omega t)$ has been omitted. Thus, in the present formulation, $\partial/\partial x = -in_0 \sin \theta_I$, $\partial/\partial y = 0$, and $\partial/\partial z = d/dz$.

When the applied magnetic field is zero, two independent problems are of particular interest. If the electric field is parallel to the plane of incidence (i.e., the x-z plane), the waves are said to be vertically polarized. It is convenient in this instance to work with (38) since the magnetic field vector will have only one component H_y:

$$H_y'' - \frac{(n*^2)'}{n*^2} H_y' + n_0^2 q^2 H_y = 0 \quad , \tag{39}$$

where $q^2(z) = n*^2(z) - S^2 = C^2 - X/U$; $S = \sin \theta_I$; $C = \cos \theta_I$. Few analytical solutions of (39) are available (see Wüster and Försterling,[16] Budden,[4] and Ginzburg[3]). The second problem arises when the magnetic field is parallel to the plane of incidence. Such waves often are said to be horizontally polarized since the electric field vector will have only one component E_y and, using (37), we find:

$$E_y'' + n_0^2 \left(C^2 - \frac{X}{U}\right) E_y = 0 \quad . \tag{40}$$

For normal incidence, $C = 1$ and (37) becomes:

$$E'' + n_0^2 \left(1 - \frac{X}{U}\right) E = 0 \quad , \tag{41}$$

-16-

where $\mathbf{E}$ can be taken to mean either of the components $\mathbf{E}_x$ or $\mathbf{E}_y$, the corresponding components of $\vec{H}$ being H_y or H_x, respectively.

For an inhomogeneous magnetoactive medium, Eq. (37) takes on the more complicated form, already noted in (22):

$$\nabla^2 \vec{E} - \nabla \nabla \cdot \vec{E} + n_0^2 \hat{K}(\omega, \vec{r}) \vec{E} = 0 \quad , \tag{42}$$

where $\hat{K}(\omega, \vec{r})$ is given in (21). A considerable simplification is derived from the assumption of a plane-layered medium in that $\hat{K} = \hat{K}(\omega, z)$. Even with the further assumption that the plane waves are normally incident on the layer, we obtain two second-order equations involving both of the unknowns $\mathbf{E}_x$ and $\mathbf{E}_y$. This system is equivalent to a fourth-order equation for either $\mathbf{E}_x$ or $\mathbf{E}_y$, in contrast to the second-order governing equation (41) for the corresponding isotropic layer problem. Exact solutions of the system of second-order equations or equivalent fourth-order equation have been found only for specific forms of $\hat{K}$.

A detailed account of the general problem outlined in the preceding paragraph is contained in what ionospheric investigators refer to as coupling theory (see, for example, Budden,[4] Chapters 18, 19, 20). The single fourth-order equation referred to above has received little attention because of its complexity. Försterling[17] introduced two new variables for the case of normal incidence and proceeded to obtain a pair of coupled second-order equations that are more readily amenable to analysis. This work was extended by Clemmow and Heading[18] to include oblique incidence. Making a change of dependent variables in the original equations (17) and (18), they derived a new set of four first-order equations that can be solved using several computational schemes or combined to form the more general pair of coupled second-order equations. Another pair of equations was used by

-17-

Gershman, Ginzburg, and Denisov[10] for the case of normal incidence with the magnetic field applied in the x-z plane. This formulation is particularly useful and, furthermore, is appropriate to principal problems of interest in this study. When $\delta = 0$ or $\pi/2$ (Fig. 1), the coupled system may be rigorously separated into two independent second-order equations. Both cases are of practical significance, particularly the former, and their solutions may be used in perturbation analyses for values of δ near zero and $\pi/2$. The basic equations are introduced below; the present preliminary study of propagation through inhomogeneous plasmas is based upon them.

For the case of normal incidence with $\vec{B}_0$ in the x-z plane, the general equations (17), (18), and (21) may be reduced to the following system of equations:

$$\left.\begin{array}{c} F_r'' + n_0^2 K_-^2 F_r = -in_0^2 BF_\ell \quad , \\[2ex] F_\ell'' + n_0^2 K_+^2 F_\ell = in_0^2 BF_r \quad , \end{array}\right\} \tag{43}$$

$$G_r = -\frac{1}{n_0} F_r' \quad , \qquad G_\ell = -\frac{1}{n_0} F_\ell' \quad , \tag{44}$$

where:

$$K_\pm^2 = 1 - \frac{X\left[Y_T^2/2 - (U - X)(U \mp Y_L)\right]}{UY_T^2 - (U - X)\left(U^2 - Y_L^2\right)} \quad , \tag{45a}$$

$$B = \frac{XY_T^2/2}{UY_T^2 - (U - X)\left(U^2 - Y_L^2\right)} \quad , \tag{45b}$$

-18-

$$F_r = E_x + iE_y \quad , \qquad F_\ell = E_x - iE_y \quad , \qquad (46)$$

$$G_r = H_x + iH_y \quad , \qquad G_\ell = -H_x + iH_y \quad . \qquad (47)$$

Two coupled second-order equations (43) specify the wave functions F_r and F_ℓ. The functions G_r and G_ℓ can be obtained from (44), after (43) has been solved; then E_x, E_y and H_x, H_y are obtained from (46) and (47), respectively. Note that for a homogeneous medium, assuming solutions of the form $F_r = F_r^{(1)} \exp(-in_0 n_{\mp}^* z)$, $F_\ell = F_\ell^{(1)} \exp(-in_0 n_{\pm}^* z)$ propagating in the positive z-direction results in two homogeneous algebraic equations for $F_r^{(1)}$ and $F_\ell^{(1)}$. Setting the determinant of the coefficients equal to zero, we obtain the Appleton-Hartree equation (29) by solving for $n_{\pm}^{*2}$ in terms of $K_{\pm}$ and B.

The fact that B vanishes identically when $Y_T = 0$ (i.e., $\delta = 0$) allows us to obtain exact solutions for a variety of electron spatial distributions. In this case, there is no coupling and (43) reduce to the following two equations for the propagation of the right- and left-handed circularly polarized waves:

$$F_r'' + n_0^2 \left(1 - \frac{X}{U - Y_L}\right) F_r = 0 \quad ,$$

$$(48)$$

$$F_\ell'' + n_0^2 \left(1 - \frac{X}{U + Y_L}\right) F_\ell = 0 \quad .$$

If we let:

$$m_r = \left(U - Y_L\right)^{-1} \quad , \qquad m_\ell = \left(U + Y_L\right)^{-1} \quad , \qquad (49)$$

then both Eqs. (48) can be written in the form:

$$F'' + n_0^2(1 - mX)F = 0 \quad , \qquad (50)$$

-19-

while both Eqs. (44) can be written:

$$G = -\frac{1}{n_0} F' \quad .$$

$$(51)$$

One might comment at this point about the nature of the polarization of the two waves propagating in a magnetoactive medium. Here we assume that the amplitudes and phase constants of E_x' and E_y' are known and determine the locus of $|\vec{E}'| = \left(E_x'^2 + E_y'^2\right)^{1/2}$ in the $z = $ constant plane, where the time dependence has been reintroduced by means of the prime in accordance with Eq. (12). Eliminating the periodic factor between the expressions for the two components, we obtain, in general, an ellipse in the x-y plane for the locus of the vector whose components are E_x' and E_y'. The wave is said to be elliptically polarized in this case. When the amplitudes of the rectangular components are equal in magnitude and their phases differ by $(2k + 1)\pi/2$, $k = 0, \pm 1, \pm 2, \ldots$, the polarization ellipse degenerates into a circle and the wave is said to be circularly polarized. It is customary to describe as right-handed circular polarization a clockwise rotation of the electric vector when viewed along the field, i.e., in the direction of propagation. Left-handed circular polarization denotes a corresponding counterclockwise rotation. If E_x' and E_y' have the same phase, the wave is linearly polarized; i.e., the locus of $\vec{E}'$ in the x-y plane reduces to a straight line. The wave polarization $\bar{\rho} = E_y'/E_x'$ is a complex number which may be conveniently introduced to show how the transverse components of $\vec{E}$ vary with time. If $\bar{\rho}$ is real, for example, E_x' and E_y' have the same phase and the wave is linearly polarized. If $\bar{\rho} = \pm i$ the polarization is circular, the minus sign identifying a clockwise rotation of the electric vector, i.e., right-handed circular polarization. If $\bar{\rho}$ is complex, the polarization is elliptical. In the present problem $(Y_T = 0)$, if we consider the F_r' wave, then F_ℓ' being independent of F_r may be taken to be zero and $E_x' = iE_y'$. Hence, $\bar{\rho} = -i$ and the wave $F_r' = E_x' + iE_y'$ is right-handed circularly polarized. By the same token, $F_\ell' = E_x' - iE_y'$ would have left-handed circular polarization.

-20-

The mathematical similarity of Eqs. (40), (41), and (50) is quite apparent. An equation of this same form appears in acoustics. Indeed it appears generally in the theory of wave propagation, another classical example of note being the Schroedinger wave equation for one-dimensional motion in quantum mechanics. It is, in fact, a second-order linear, homogeneous, ordinary differential equation with variable coefficients and arises often in mathematical physics. The generalized Lamé equation (see Whittaker and Watson[19]) encompasses a large class of such equations, and by a suitable treatment of the singularities one can arrive at the equations of Mathieu, Legendre, Bessel, Weber, Stokes, etc. Since no solution can be written in terms of known functions for an arbitrary functional form of the index of refraction, particular cases in which this can be done acquire great interest. Such solutions can be obtained when the propagation equation is reducible to one of the aforementioned standard forms. In general, when this is not the case, it is necessary to introduce the WKB approximation, when applicable, seek asymptotic solutions, or employ numerical methods. Solutions may be directly obtained when the WKB approximation is valid. This procedure is briefly reviewed, therefore, at the end of the section. Expansion procedures have been employed successfully in a number of instances while high-speed computing machines may be used to implement the several numerical techniques available. More specific reference to these approximate procedures will be made following the discussion of the WKB method.

In the following summary we will review the known exact solutions for the cases of: (i) horizontally polarized waves, oblique incidence, no magnetic field [Eq. (40)]; (ii) normal incidence, arbitrary polarization [Eq. (41)]; and (iii) circularly polarized waves, normal incidence, normal uniform applied magnetic field [Eq. (50)]. The primary purpose of the review is to outline the solutions for the wave forms propagating in inhomogeneous plasmas. The determination of reflection and transmission coefficients will depend then on the nature of the specific boundary value problems of interest. This will involve the solutions for the wave forms propagating in each medium, characterized by its index of refraction, and the requirement that F, for

-21-

example, and F' [in view of (51)] be continuous at the interface that separates two media. If the inhomogeneous medium (z > 0) extends to infinity, it is necessary to disregard the waves that propagate in the negative z-direction determined from the asymptotic behavior of the solution for large z. This boundary condition is based on the physical reason that no source of energy exists at infinity. Reference will also be made in the summary to detailed solutions of boundary value problems which have been obtained.

Exact solutions of the propagation problems described above have been obtained for various assumed functional forms of the index of refraction itself as well as separately for $X(z)$ with Z = constant and, in a few cases, $Z(z)$ with X = constant. In general, the utility of these formal solutions is limited by the difficulty in abstracting numerical results since the arguments and/or orders of the special functions involved are complex numbers. Considerable information can be obtained, however, from the limiting forms of the solution for small and large values of the arguments. Although extensive quantitative results also can be obtained to any desired accuracy by solving the equation numerically, analytical expressions are particularly useful, if available, for the general analysis of propagation in inhomogeneous plasmas. Further, it should be remarked that although a surprisingly large number of exact solutions have been derived, a detailed quantitative analysis of the reflection and transmission coefficients and of their specific dependence on the physical parameters of the problem is given in only a few papers. Recent interest in transmission problems arising in space programs and laboratory plasma studies has resulted in a number of papers which provide such information with varying amounts of detail. There is often no reference in these papers to previously obtained general solutions for the inhomogeneous plasma and the corresponding related, if not equivalent, boundary value problem. This is due, in part, to the fact that many of these solutions were presented in the magneto-ionic literature. Indeed, our present purpose is to review the various known solutions for possible use in several different fields. In this

-22-

regard, it is recognized that the appropriate geometry may differ markedly for different applications. In ionospheric work, for example, propagation across an inhomogeneous layer separating free space and a homogeneous medium is analyzed with principle emphasis given to the study of reflection. In the plasma physics applications noted above, both bounding regions are often assumed to be free space and both reflection and transmission are of interest.

The function $w(z)$ will be used to represent the unknowns E_y, E, or F in Eqs. (40), (41), and (50), respectively. Hence:

$$w'' + n_0^2 q^2 w = 0 \quad , \quad q^2(z) = n^{*2}(z) - S^2 = C^2 - mX \quad , \quad (52)$$

where m is defined in Eq. (49), either with or without an applied magnetic field, and $C = 1$ (normal incidence) in the magnetoactive case. The inhomogeneous plasma solutions, therefore, will apply to all three cases on this basis. Reflection and transmission coefficients (referred to as R and T, respectively, in the outline) are also applicable to each problem even though the reference work may have been concerned with one case in particular. For this purpose, it is convenient to assume that the incident wave is linearly polarized along the x-axis. Unless otherwise stated Z is taken to be constant.

1.   Linear: $X(z) = a(z - z_0)$.

Substitution, $\xi = -C^2(n_0/am)^{2/3}[1 - (am/C^2)(z - z_0)]$, results in Stokes' equation, $d^2 w/d\xi^2 = \xi w$. General solution expressed in terms of Airy functions, $w = c^{(1)}\mathcal{A}i(\xi) + c^{(2)}\mathcal{B}i(\xi)$. Alternatively, let $(2/3)\xi^{3/2} = i\zeta$, $w = \zeta^{1/3}v$, to obtain Bessel functions for the solution, $v = c^{(1)}J_{1/3}(\zeta) + c^{(2)}J_{-1/3}(\zeta)$. The basic inhomogeneous plasma solution appears in a number of references along with expressions for R and T for the simplest boundary value problem of interest defined in Fig. 2a. Problems 2b and 2c were considered by Hartree[20] for $Y = 0$, with particular attention given to R. Recently, detailed calculations of R and T have been made for 2b through 2e by Albini and Jahn[21] assuming $Y = 0$, $C = 1$. Hermann[22]

-23-

considered 2a for $Y = 0$, $C = 1$ and derived a useful approximate procedure. Taylor[23] determined R for 2b using a formulation based on the transmission line equations, but detailed numerical results were not presented.

      2.   Square law: $X(z) = a(z - z_0)^2$.

              Substitution, $\xi = (4n_0^2 am)^{1/4}(z - z_0)$, $k + 1/2 = (4n_0^2 am)^{-1/2}n_0^2 C^2$,

results in Weber's equation, $d^2 w/d\xi^2 + (k + 1/2 - \xi^2)w = 0$, and the general solution, $w = c^{(1)}D_k(\xi) + c^{(2)}D_{-k-1}(\xi)$. Problem 2f has been considered by Hartree,[24] Wilkes,[25] and Rydbeck.[26]

      3.   Parabolic: $X(z) = X_1\left[1 - (z - z_1)^2/z_0^2\right]$, $|z - z_1| \leq z_0$.

              Substitution, $\xi = (-4n_0^2 mX_1/z_0^2)^{1/4}(z - z_1)$,

$k + 1/2 = (-4n_0^2 mX_1/z_0^2)^{-1/2}n_0^2(C^2 - mX_1)$, results in Weber's equation (ibid.). Problem 2g was considered by Rydbeck[27] and Pfister.[28]

      4.   Exponential: $X(z) = \exp(az)$.

              Substitution, $\xi = (2n_0 m^{1/2}/a)\exp[(az + i\pi)/2]$, $k = 2in_0 C/a$,

results in Bessel's equation, $\xi^2 d^2 w/d\xi^2 + \xi dw/d\xi + (\xi^2 - k^2)w = 0$, and the general solution, $w = c^{(1)}J_k(\xi) + c^{(2)}J_{-k}(\xi)$. This problem has been examined by a number of authors. Stanley[29] obtained the solution for ionospheric propagation as a limiting case of Epstein's[30] method. The following distribution is more appropriate to many applications in plasma physics and is considered later in this report: $X(z) = X_\infty + (X_0 - X_\infty)\exp(-az)$, $z > 0(a > 0)$.

      5.   Trigonometric: $X(z) = (X_1/2)[1 + \cos(\pi/z_0)(z - z_1)]$, $|z - z_1| \leq z_0$. Change of independent variable gives Mathieu's equation; however, a numerical solution of 2h can be used conveniently to compute values of $|R|$, as noted by Budden.[4] Unlike (3), $X'(z)$ is continuous at $|z - z_1| = z_0$. This is found to be useful in ionospheric applications.

6. Hypergeometric: $X(z) = 1 - m^{-1}\{S^2 + \epsilon_1 + e^{\xi}(e^{\xi} + 1)^{-2} \times [(\epsilon_2 - \epsilon_1)(e^{\xi} + 1) + \epsilon_3]\}, \xi = (z/a) + \beta, \epsilon_1 = -(c - 1)^2/4n_0^2 a^2, \epsilon_2 = -(a - b)^2/4n_0^2 a^2, \epsilon_3 = (a + b - c + 1)(a + b - c - 1)/4n_0^2 a^2.$

General solution is expressed in terms of hypergeometric functions. A number of interesting distributions, known as Epstein profiles, are included in this case. The details are given in Epstein's[30] classical paper for an isotropic medium. Of particular note also is the useful procedure for obtaining reflection and transmission coefficients for an Epstein layer. The following special cases are generally singled out:

a. $X(\xi) = m^{-1}\{C^2 - 1/2(\epsilon_2 + C^2) - (1/2)(\epsilon_2 - C^2) \tanh[(z - z_1)/2a]\}$.

b. $X(\xi) = -(1/4m)\epsilon_3 \operatorname{sech}^2[(z - z_1)/2a].$

7. Inverse square law: $X(z) = z^{-2}.$

Transformation, $w = z^{1/2}v$, results in Bessel's equation and the general solution, $w = c^{(1)}z^{1/2}J_p(n_0Cz) + c^{(2)}z^{1/2}J_{-p}(n_0Cz), p = [(1/4) + n_0^2 m]^{1/2}.$ Taylor[31] examined this problem, assuming the incident wave to be propagating from $z = +\infty$. The asymptotic expansion of J is used to satisfy the boundary condition at $+\infty$ and to determine R. The more customary problem in which the wave is incident at $z = 0$ propagating into the plasma region $z > 0$, where $X(z) = (z + a)^{-2}$, $a > 0$, is also of interest. The inhomogeneous plasma solution given above may be modified readily in this case and the appropriate boundary conditions may be applied.

8. Polynomial: $X(z) = \sum_{i=0}^{k} a_i z^i$

General infinite series solution for $E_y(z)$ was formally obtained by Taylor[32] with primary attention focused on R. Specific consideration given to quadratic, $X(z) = a_1 z + a_2 z^2$.

9.   Periodic: $X(z) = X(z + a)$.

Expand X into a Fourier series and seek Fourier series solution for w. Vassiliadis[33] formulated the general procedure for determining the coefficients in the series for the electric field and examined specifically $X = (X_0/2)(1 + \cos 2\pi z/a)$, but did not consider a particular boundary value problem. His simple variational approach in obtaining solutions is also of interest.

10.   Generalized power law (normal incidence): $n*^2(z) = a^2(1 + az)^\ell$, $\ell \neq -2$. The general solution can be written in the form, $F = (1 + az)^{1/2}v$, where v is a Bessel function of order p and argument ξ, $p^2 = (\ell + 2)^{-2}$, $\xi = 2n_0(ap/a)(1 + az)^{(\ell + 2)/2}$. The appropriate form of the general solution will depend on the boundary value problem. To satisfy the condition at $z = +\infty$ in the semi-infinite case (assuming the imaginary part of a to be negative), the assumption $\ell > -2$ would require the use of the Hankel function of the second kind while $\ell < -2$ would require the Bessel function of the first kind of order $p = -(\ell + 2)^{-1}$. A number of such solutions (for $\ell = -1$, the expression for $n*^2$ may contain an additive constant) have appeared in the ionospheric literature (see Ginzburg[3]).

11.   Exceptional case, $\ell = -2$: $n*^2(z) = c(b + z)^{-2}$.

This case is of particular interest because the solution can be expressed in terms of elementary functions, $F = c^{(1)}(b + z)^{r_1} + c^{(2)}(b + z)^{r_2}$, $r_{1,2} = 1/2 \pm (1/4 - n_0^2 C)^{1/2}$. Ginzburg points out the following solution to problem 2i: $R = (i/2)(n_0a + \beta)^{-1}$, $\beta = [(n_0a)^2 - 1/4]^{1/2}$, $n_0a > 1/2$. In the WKB limit, $n_0 a \gg 1$ and $\beta \approx n_0a$ such that $R = i/4n_0a = i\lambda_0/8\pi a = -(i\lambda_0/8\pi)(d/dz)(n*^2)|_{z=a}$, evaluated in the plasma.

12. $n*^2(z) = (az + b)/(cz + d)^5$, normal incidence.

The general solution was obtained by Penico[34]:
$$F = (cz + d)\left\{c^{(1)}\text{Ai}[-D^{-2/3}(az + b)/(cz + d)] + c^{(2)}\text{Bi}[-D^{-2/3}(az + b)/cz + d)]\right\},$$
where $D = ad - bc$. No particular boundary value problem was considered.

13.   Variable collision frequency, constant X.

A few solutions have been obtained. In each case, Z was assumed to be a decreasing function of z, and very little quantitative analysis of the effect on propagation is presented.

a.   (-1) power law: $Z(z) = a/z$.

Change of independent variable results in Gauss' equation with the general solution in terms of confluent hyper-geometric functions. Wilkes[25] examined this problem briefly.

b.   Exponential: $Z(z) = a \exp(bz)(b < 0)$.

This distribution has been derived from the Epstein profiles.

The exact solutions summarized above were used by the referenced authors to analyze propagation in inhomogeneous plasmas for various applications. Often these analyses were intended to provide a general examination of the problem and only limited quantitative information was obtained. As we have already noted, even if the assumed form of the index of refraction is appropriate to a specific problem of interest, detailed numerical results are difficult to obtain from the formal solutions without resorting to the use of high-speed computers. In addition, it may be necessary to contend with more general forms of the index of refraction. Various approximate procedures and analyses, which apply to these situations as well as to more complicated propagation equations than (52), will be discussed briefly in the remainder of this section.

The WKB analysis is a particularly useful means of obtaining approximate solutions. To demonstrate the mathematical procedure, as well as complete the discussion of the physical significance given in the preceding section, we shall now consider the WKB analysis of Eq. (52). The substitution $w = \exp(\int v dz)$ gives rise to a nonlinear Riccati equation for $v(z)$. Assuming that $q^2(z)$ is a slowly varying function, we can derive the following

approximation to the general solution of (52):

$$w = c^{(1)} q^{-1/2} e^{-in_0 \int q dz} + c^{(2)} q^{-1/2} e^{in_0 \int q dz} \qquad (53)$$

valid over an interval for which:

$$\frac{1}{n_0^2} \left| \frac{3}{4} \left(\frac{q'}{q^2} \right)^2 - \frac{q''}{2q^3} \right| \ll 1 \qquad . \qquad (54)$$

The corresponding solution for G, for example, would be obtained from (51). The "slowly varying" character of the plasma properties and the so-called reflection points are formally specified by (54) in that q' and q'' must be sufficiently small while q cannot be too small.

The WKB solution (53) can be employed as a mathematical expression of ray theory in the following way: The governing equations for w and u [where the function u(z) represents H_x, H, or G corresponding to the values E_y, E, and F for w] are obtained from (17) and (18):

$$w' = -n_0 u \qquad , \qquad u' = n_0 q^2 w \qquad . \qquad (55, 56)$$

Separate w and u into two parts, $w = w^{(1)} + w^{(2)}$ and $u = u^{(1)} + u^{(2)}$, the first traveling in the positive z-direction, the second, in the negative z-direction. Using the characteristic exponential form of the solution for a homogeneous medium for u, we reduce (56) to $u^{(1)} = iqw^{(1)}$, $u^{(2)} = -iqw^{(2)}$. Substituting these expressions into (55), we obtain two coupled second-order equations in $w^{(1)}$ and $w^{(2)}$. The first approximation to the solution of this system neglects the coupling, and (53) follows from the resultant two independent equations for $w^{(1)}$ and $w^{(2)}$. Whenever the index of refraction is varying locally, a reflection process is taking place in which one wave can generate some of the other as shown by the coupling in the two equations for

-28-

$w^{(1)}$ and $w^{(2)}$. The physical processes described at the conclusion of Section III are thereby clarified mathematically, and the resultant formal solution and limitation are expressed by Eqs. (53) and (54), respectively. It should be noted that the coupling process referred to in this discussion is between the waves (either F_r or F_ℓ in the magnetoactive case) that are propagating in the positive and negative z-directions. This is quite apart from the so-called coupling theory in magneto-ionics between the right- and left-hand waves propagating in one direction. The solution of (50) for F_r and F_ℓ separately or of the more complicated Eqs. (43), in which both waves are coupled, can be obtained by means of the WKB approximation, if applicable. The WKB analysis described above adequately provides the mathematical and physical considerations required in this study. More general presentations of this classical subject are readily available in the literature (see Ginzburg, [3] for example).

The WKB method and the related phase integral method which applies in the vicinity of reflection points (see Budden, [4] Chapter 20, for example) have been employed in the analysis of a variety of problems and applications. The numerical evaluation of such solutions or, as we have previously noted, of existing exact solutions has been considerably simplified by the advent of high-speed computers. Indeed, in this way the solution of even more general propagation problems has been realized, although analytical expressions and their inherent advantages are not provided. Budden[4] outlines the general principles underlying the computational procedure for integrating the ordinary differential equations that govern the reflection of a plane radio wave from a horizontally stratified (magnetoactive) ionosphere. The numerical analysis of boundary value problems of present interest is considered in several mathematics references. In the following discussion, reference is made to several papers in which specific calculation procedures are set forth. The principal purpose of this discussion is to take note of analyses of Eqs. (40), (41), or (50) for more general forms of the index of refraction than were considered above as well as work which has been done in connection with Eq. (39).

Following this, brief consideration will be given to several approximate mathematical procedures which, along with the WKB method, have been used to analyze related but somewhat more general propagation problems.

Tyras and Held[35] considered the problem of normal incidence into a stratified, lossy, magnetoactive plasma that is assumed to consist of a number of homogeneous layers of arbitrary thickness. No particular pattern for the electron density and collision frequency variations is assumed. Both the cases $\delta = 0$ and $\delta = \pi/2$ are examined, and the results are left in general form, suitable for high-speed computer calculations. Once programmed, numerical results can be obtained quickly. A representative model of a re-entry type of inhomogeneous plasma was assumed by Harley and Tyras[36] who used this numerical procedure to make illustrative transmission calculations. The primary purpose in using a numerical approach is to obtain detailed quantitative information for a specific problem including complicated variations of $N(z)$ and $\omega_c(z)$. The general effect of the several parameters of the problem on transmission is not shown due to the limited number of cases. It is found, however, that the best improvement in transmission is obtained when the magnetic field is applied in the direction of propagation (as would be expected) and the incident wave is circularly polarized.

Several methods have been proposed for solving the general problem of plane wave propagation through a plane-layered medium. Reference was made above (see also Levy,[37] for example) to detailed analyses in which the plasma was assumed to consist of a stack of homogeneous layers. In this case, the well-known matrix method or sandwich formulas can be used. The problem can also be formulated as an integral equation and approximate solutions obtained by iteration. Variational techniques have been derived, and the Riccati equation formulation has been considered. It would appear to this author, however, that the differential equations are suitable for standard numerical analysis procedures when extensive quantitative results are desired. Thus, the field intensity can be obtained by step-by-step numerical integration from which the reflection and transmission coefficients are readily determined.

-30-

Richmond,[38] for example, outlined this technique both for problems of normal and oblique incidence ($B_0 = 0$). Although he considered inhomogeneous, dielectric, radomes (the application is itself worth mentioning in this review), the detailed procedure clearly is applicable to the present problem of lossy plasmas. Excellent agreement between approximate and exact solutions was obtained for the assumptions of constant, linear, and exponential variations in the properties of the medium. Klein, et al.,[39] treated this same problem for the case of a semi-infinite plasma. The normal incidence problem was solved by numerical integration in the region of rapid variation of electron density and by the WKB method outside this region. Computations were made for a large number of cases where the electron density distribution was assumed to be exponential, for the sake of convenience. A numerical procedure was employed to solve the problem of oblique incidence (provided the incident wave is not too close to normal incidence) and calculations were made assuming Z and λ_o small. It should be remarked that this analysis was not restricted to the problem of horizontal polarization only.

We have discussed at some length in this section the problem of finding solutions to Eq. (52) and, as a result, to a number of propagation problems of interest. For certain forms of the function $q^2(z)$, (52) can be transformed into one of the classical equations of mathematical physics, and exact solutions were said to have been obtained in the sense that they were expressible in terms of known functions. One might well refer to the general theory of series solutions of (52), in terms of the behavior of $q^2(z)$, which includes these special functions, to formalize the mathematical analysis. The numerical analysis of this equation for the same or more complicated forms of the function is also exact in that any desired accuracy can be obtained for the solution without restricting the parameters of the problem. No analytical expression would be available. By the same token, more general problems than (52) could, in principle, also be solved numerically. The over-all evaluation and interpretation of an even greater amount of data and the inherent difficulty of the appropriate equations would limit this approach to restrictive

-31-

cases of specific interest. The WKB and phase integral methods have been employed to obtain approximate solutions for more general problems of interest. Perturbation solutions of Eq. (43) have been examined for values of δ which are close to 0 or $\pi/2$ (see Ginzburg[3]). From the literature on coupling theory, we find that solutions of the Försterling equations have been obtained by successive approximations (see, for example, Gibbons and Nertney[40,41]). Budden and Clemmow[42] used the formulation proposed by Clemmow and Heading[18] to obtain solutions; in particular, the system of first-order equations was derived for the cases of normal incidence, oblique magnetic field and oblique incidence, normal magnetic field and approximate solutions obtained assuming the coupling terms small. Recently, Cohen[43] extended coupling theory to nonuniform, plane-layered, magnetoactive media. WKB solutions were obtained for the two limiting cases of negligible and strong coupling, as defined by Försterling, assuming that X, Y, Z << 1. Note that solutions often can be obtained by appropriately restricting the range of the three characteristic frequency ratios. Several interesting examples of such analyses are, Heading,[44,45] Poverlein,[46] Wilkes,[47] and Heading and Whipple.[48] The references noted in this paragraph illustrate, in a rather abbreviated manner, procedures used and results obtained for more general propagation problems than (52). We have focused our primary attention, in this report, on those problems which are governed by Eq. (52).

SECTION V

THE EFFECT OF AN APPLIED MAGNETIC FIELD ON PROPAGATION
THROUGH A HOMOGENEOUS PLASMA SLAB

The initial quantitative treatment of the effect of an applied magnetic field on propagation in re-entry communication problems has been made by various authors using the simplified model of a homogeneous plasma. An indication of this effect may be obtained from the complex index of refraction as was noted in the paragraph following Eq. (31). Some results were obtained on this basis in the early investigation of Bachynski, Shkarofsky, and Johnston,[49, 50] for example. A somewhat better estimate of the problem can be obtained from the consideration of propagation across a single interface. Numerical results were obtained for this semi-infinite case assuming normal incidence and normal applied field by, for example, Bachynski, Johnston, and Shkarofsky,[51] Fante,[52] and Hodara.[53] French, Cloutier, and Bachynski[54] studied some general aspects of the problem with the magnetic field both parallel and normal to the propagation direction. Their observations on the existence of very low-frequency "windows" due to the ion cyclotron mode represent a possible application which is still to be explored and merits additional consideration. Graf and Bachynski[55] included in their analysis the effect of the orientation of $\vec{B}_0$ and the polarization of the incident wave.

Additional results of a similar nature have been published both in company reports and in journals. In this section, we shall consider a homogeneous plasma slab, assuming the incident wave and applied magnetic field to be normal to the slab. Detailed parametric calculations were made to determine the combined effect of Y, X, Z, and L/λ_0 on transmission. Specific exemplary flight conditions were examined on this basis also, as was the initial consideration of a proposed experimental shock-tube program.

-33-

A. General Solution and Discussion of Parametric Results

Consider a homogeneous plasma slab of thickness L separating the free space regions $z < 0$ and $z > L$. A linearly polarized (say, along the x-axis) plane wave is normally incident on the abrupt free space-plasma boundary $z = 0$, and a uniform magnetic field is applied in the direction of propagation. The solution to the governing equation (50) for the propagation of both the right- and left-handed circularly polarized waves in the plasma region $0 < z < L$ is given by:

$$F(z) = c^{(3)} e^{-in_0 n^* z} + c^{(4)} e^{in_0 n^* z} \quad , \quad n^* = (1 - mX)^{1/2} \quad . \quad (57)$$

In the free space regions $z < 0$ and $z > L$, respectively:

$$F = c^{(1)} e^{-in_0 z} + c^{(2)} e^{in_0 z} \quad , \quad F = c^{(5)} e^{-in_0 z} \quad . \quad (58, 59)$$

Continuity of F and F' [in view of (51)] at the two interfaces $z = 0$ and $z = L$ leads to four equations for the amplitudes $c^{(1)}$ through $c^{(5)}$ from which the following expressions for the reflection and transmission coefficients can be derived:

$$R = \frac{c^{(2)}}{c^{(1)}} = \frac{(n^{*2} - 1) \exp(-in_0 n^* L) + (1 - n^{*2}) \exp(in_0 n^* L)}{(n^* + 1)^2 \exp(in_0 n^* L) - (n^* - 1)^2 \exp(-in_0 n^* L)} \quad , \quad (60)$$

$$T = \frac{c^{(5)}}{c^{(1)}} = \frac{4n^* \exp(in_0 L)}{(n^* + 1)^2 \exp(in_0 n^* L) - (n^* - 1)^2 \exp(-in_0 n^* L)} \quad . \quad (61)$$

Separate results are obtained for the right- and left-hand waves, using the appropriate value of n^*. Evaluating the real part of the complex Poynting

vector in the free space regions $z > L$ and $z < 0$, we obtain the following expressions for the ratio of the transmitted and reflected energy to the incident energy:

$$\epsilon_T = \frac{1}{2}\left(|T_r|^2 + |T_\ell|^2\right) \quad , \qquad \epsilon_R = \frac{1}{2}\left(|R_r|^2 + |R_\ell|^2\right) \quad . \qquad (62, 63)$$

A detailed calculation of the reflection and transmission coefficients (magnitude and phase) for both waves has been made covering a wide range of the three frequency ratios and the ratio L/λ_0. We will be interested primarily in transmission for the parametric analysis of the problem in this subsection. The degradation in transmitted energy:

$$db_T = 10 \log_{10} \epsilon_T \quad , \qquad (64)$$

was obtained for $L/\lambda_0 = 0.5, 1, 2, 4$ with $0.01 \leq Z \leq 1$, $0.64 \leq X \leq 400$, and $0 \leq Y \leq 100$. From the graphs of db_T versus Y (Figs. 3 and 4 are included for illustrative purposes), we can determine useful crossplots of X versus Y at constant values of db_T, say, -15, -10, and -5, with Z and L/λ_0 as parameters. The dependence of db_T on the applied magnetic field, which will, of course, be affected by the values of the parameters X, Z, and L/λ_0, can be observed first in Fig. 3a. It is immediately apparent that when X is of order one (in particular, note $X = 1.21, 1.44, 2.25$, and, to a lesser extent, 4) db_T is strongly dependent on Y. This is indicative of the significant improvement in transmission that can be derived for relatively small values of Y. The dependence is much less pronounced for larger values of X, especially as the magnitude of db_T is decreased. Qualitatively similar results are shown in Fig. 4a; however, it is clear that the increase in Z generally decreases the favorable effect of Y. These observations will be discussed in greater detail below.

The principal factors that were introduced above can be conveniently examined in some detail from the constant db_T curves of X versus Y shown in

-35-

Figs. 5 through 16. To elaborate further on the previous general observations, we first consider Figs. 8, 9, and 10 for $L/\lambda_0 = 1$. In the latter figure ($Z = 1$) for $X = 1.44$; $db_T = -15$, -10, and -5 is obtained when $Y = 0.95$, 1.6, and 2.75, respectively. With telemetry frequencies ($f = 240$ Mc), field strengths of the order of several hundred gauss are involved, and even in the microwave region (say $f = 1$ kMc), the maximum value of 1000 gauss corresponding to 5 db is still quite practical. Note that a 68-percent increase in field strength is required to improve transmission from 15 to 10 db, while an additional increase of 72 percent would result in 5 db. As one would expect, larger values of Y are required when the electron density of the plasma is increased. Thus, for $X = 100$, corresponding values of $Y = 10.5$, 16, and 41 are obtained. The magnetic field strengths are an order of magnitude larger than in the preceding example, a maximum of 14,600 gauss being needed to obtain 5 db for the microwave application. Although a 56-percent increase in Y will reduce the loss from 15 db to 10, a further increase of 256 percent would be required to go from 10 to 5 db. (Corresponding values of 83 and 326 percent are obtained when $X = 400$.) Decreasing the collision frequency by an order of magnitude to $Z = 0.1$ (Fig. 9) resulted in a reduction of the magnetic field required by as much as a factor of 3. Although the required magnetic field generally is still lower for $Z = 0.01$ (Fig. 8), the resonant conditions are quite pronounced in this case, and the preceding trends are substantially modified in certain instances. Increasing the slab thickness would, of course, degrade the transmitted signal. A uniform increase in Y over the above results is obtained at $L/\lambda_0 = 4$, for $Z = 1$, as shown in Fig. 16. The combined effect of the several parameters is complicated by the aforementioned resonant conditions in this case for $Z = 0.1$ and 0.01; detailed results are given in Figs. 15 and 14, respectively. In Figs. 5, 6, and 7 $L/\lambda_0 = 0.5$ is considered, while, to a lesser extent, $L/\lambda_0 = 2$ is considered in Figs. 11, 12, and 13.

A substantial improvement in transmission can be derived, in many cases, through the use of relatively small magnetic fields. Clearly, as we

have seen, this depends on the values of L/λ_0, X, Z, and f involved. Of particular significance, however, is the benefit still to be derived from the application of reasonable fields if one accepts as great as 10 to 20 db degradation in the transmitted signal, this in contrast to the requirement of an order of magnitude larger field for the same condition when near complete transmission is sought (as was done in a number of early rough estimates). This is due to the fact that the variation of db_T with Y is considerably more gradual for lower values of db_T, particularly when $X >> 1$. In this regard, it should be observed that particular attention must be given to such effects as inhomogeneities and nonuniformities if improved estimates of db_T are to be obtained since the difference in the resultant prediction of the required field may be significant.

B.   Evaluation of Several Re-entry Applications

The homogeneous plasma slab boundary value problem outlined above was used also to estimate transmission in several exemplary re-entry situations.   The effect of an applied magnetic field is determined thereby for particular applications of interest.   In the first case, a 10-degree half-angle cone was considered at zero angle of attack with the flight conditions shown in Fig. 17.   An equilibrium boundary layer analysis was made to determine the maximum enthalpy and temperature, from which the plasma properties were obtained.   We computed the boundary layer or slab thickness using the inviscid flow properties, assuming the actual cone wall temperature to be 1500°K and the axial length, 11 feet.   The results are shown in Fig. 18. Despite the fact that the plasma is highly overdense in most of the cases, transmission is obtained due to the correspondingly small values of L/λ_0 involved.   The determination of transmission for each flight condition is greatly dependent on this "skin-depth" effect.

Before discussing the present results, a reference level of acceptable transmission must be established.   Since this value is often difficult to ascertain, even for current applications which are well along in the development phase, it will be necessary to base our remarks on several possible

-37-

assumed values. We shall consider below the telemetry frequency, $f = 240$ Mc, assuming re-entry velocities of 26,000 fps (Fig. 19) and 23,600 fps (Fig. 20) and the microwave frequency, $f = 3$ kMc, with $u = 26,000$ fps (Fig. 21). If 10 db degradation in transmitted energy is acceptable, no magnetic field is required in the second case (Fig. 20) since the maximum loss is 9.75 db at 150,000 feet. If, however, a maximum of 5 db is allowed, then "blackout" should occur at both 100,000 feet and 150,000 feet. Although the 150,000-foot condition is more critical than the 100,000 with no magnetic field, we should take note of the particularly strong dependence of db_T on the applied magnetic field in the former case. Thus, 250 gauss is required to eliminate "blackout" at 150,000 feet while 600 gauss is needed at 100,000 feet. The more severe re-entry condition with $u = 26,000$ fps (240 Mc) is shown to be qualitatively similar in Fig. 19. At 150,000 feet, $db_T = -25.6$ for $B_0 = 0$; 600, 1200, and 2650 gauss would be required to obtain -15, -10 and -5, respectively. At 100,000 feet, $db_T = -16.8$ for $B_0 = 0$, whereas larger field strengths of 1200, 3600, and 7800 gauss now are required. If the allowable loss is reduced farther so that the magnitude of db_T is 4 or less, the 50,000-foot condition would be critical and even larger fields would be required since the corresponding curve is more gradual than the 100,000-foot case. Comparative values for $f = 3 \times 10^9$ cps are given in Fig. 21.

The favorable effect of an applied magnetic field on re-entry communication is shown for a number of practical conditions of interest in the tabulated values of Fig. 18 along with Figs. 19, 20, and 21. A preliminary analysis of the boundary layer region on a 10-degree cone clearly indicates the existence of electron density gradients in the direction of propagation. The magnetic field required to provide a specified acceptable level of transmission can be reduced substantially when the more realistic inhomogeneous plasma is included; this will be examined later in this report.

Blunt body-stagnation point calculations were also made for several typical re-entry conditions. The assumption of a homogeneous plasma slab is quite appropriate in this instance. The stagnation condition gives rise to

-38-

large values of the plasma and collision frequency. In the extreme case considered, f_p is of the order 10^{12} cps and f_c of the order 10^{12} to 10^{13} cps. With no applied magnetic field, losses in excess of 30 db are obtained with frequencies of 240 Mc to 30 kMc. Since X, Z >> Y in this case, even for $B_0 = 100,000$ gauss, the magnetic field is of no practical value. When the maximum values of f_p and f_c are of the order of 10^{11} cps, transmission can be increased to "satisfactory" levels; however, field strengths of the order of 50,000 gauss would be required. The practical use of a magnetic field becomes more apparent when $f_p < 5 \times 10^{10}$ cps. For f = 30 kMc significant improvement in transmission is derived from the application of 5 to 10,000 gauss while 25,000 gauss is required for this purpose when f = 240 Mc. The discussion is intended to be qualitative in this case and to bracket the general results of interest inasmuch as the use of blunted re-entry vehicles has been reduced greatly in recent years.

C. Formulation of a Meaningful Experimental Program

The immediate purpose of an experimental program for the present problem is essentially three-fold: to obtain quantitative results on the effect of an applied magnetic field on propagation, to correlate these results with theory to the extent possible, and to investigate related problem areas suggested by the previous theoretical considerations. The end purpose is to enable prediction of propagation for several applications of interest. In the re-entry cone application, for example, the functional form of the electron density distribution may be complicated and variations in collision frequency may be of importance. The theoretical problem may be complicated further by the requirement of oblique incidence, the practical existence of a nonuniform, applied (as well as induced), magnetic field, and transverse gradients of the plasma properties. In view of the intractable nature of such a problem, as well as our inability to accurately measure or compute the plasma properties, it would seem that an experimental flight program would be of limited general value. Thus, specific information could be obtained for a set of actual flight conditions, but, the results could not be generally

-39-

interpreted so as to evaluate theoretically some other flight case. It would seem reasonable, therefore, to attempt a definitive controlled laboratory experiment. An idealized experiment can be formulated for which analysis is possible. General information can be obtained from such a correlation. However, theoretical estimates of an actual flight problem may still require additional laboratory experiments and analyses on the several complex conditions noted above.

There are several ways in which the proposed laboratory experiment can be approached. Related work has been done in the field of plasma diagnostics by Jahn[56,57] and Betchov and Fuhs,[58] to name a few. The interaction of microwaves with plasmas has been studied experimentally from several points of view by Daiber and Glick,[59] Rao, Verdeyen, and Goldstein,[60] and Jones and Gould.[61] The transmission problem has been examined to some extent by Lueg and Wood,[62] Rothmann and Morita,[63] and Delosh,[64] as well as in several company proposals. The shock tube has been used in the study of plasma diagnostics. In addition, shock-tube techniques and theory have been developed to a point where the generated plasma is well known. With this basic prerequisite for the formulation of a meaningful experimental propagation study, the shock tube is a good choice for the plasma source. It also enables one to employ a representative analytical model which can be solved to interpret the experimental results.

The homogeneous plasma slab solution can be used as a guide now in the initial evaluation of a possible shock-tube program. The underlying physical model is appropriate for the present purpose, and preliminary calculations of the boundary layer indicate that the inhomogeneous region is small. Properties of shock-tube-generated plasmas (in argon) were computed using Refs. 65 and 66, and are shown in Figs. 22 and 23. It should be noted that although the collision cross sections of both the neutral atoms and the positive ions were included in the computation of collision frequency, the author has not established that the reference work represents the latest information regarding this difficult calculation. The data shown will be satisfactory, however, for the present purpose.

Both the plasma and collision frequency can be varied by several orders of magnitude for the range of Mach numbers (6 to 9) and pressures shown. Although the desired values of plasma conditions are thus readily obtained, the futility of even contemplating such an experimental program without the benefit of some preliminary analysis may be demonstrated easily. It is not our purpose at this time to consider the problems involved in any detail but rather to employ reasonable theoretical results to aid initial thinking along these lines. On this basis, the homogeneous plasma slab analysis may be used to point out the following useful results. It is desirable when working with microwave receivers, in general, to restrict both the transmitted and reflected power levels as follows:

$$0.05 \leq \epsilon_T \leq 0.95 \qquad , \qquad 0.05 \leq \epsilon_R \leq 0.95 \qquad . \qquad (65)$$

At the same time, good experimental procedure dictates that neither very small nor very large gradients in these quantities be permitted. Hence, we will also stipulate that:

$$\frac{d\epsilon_T}{dX} \qquad , \qquad \frac{d\epsilon_R}{dX} \geq 0.05 \qquad . \qquad (66)$$

In this manner, a minimum variation is assured for the purpose of inter-preting the experimental results. Excessively large gradients (for example, near-resonant conditions) which would place an unnecessary burden on the results in view of the expected errors and uncertainties of the problem are also ruled out. Imposing these constraints on the numerical evaluation of Eqs. (60) and (61), we obtain the results shown in Figs. 24 through 37.

Figures 24 and 25 show plots of Z versus X for Y = 0 and 0.5, respectively, with $L/\lambda_0 = 4$. The corresponding shock-tube data are presented, assuming f = 24 kMc, for an initial pressure of 1, 5, and 15 cm and parametric values of the shock Mach number. Transmission and

-41-

reflection data are not required simultaneously for diagnostic purposes, although the additional information thus provided is of definite value. Both results are of interest, however, in the study of propagation, with particular emphasis given to the transmission. The acceptable regions defined by (65) and (66) are clearly shown in these graphs. The largest region of overlap in which both transmission and reflection data can be obtained occurs when $p_1 = 1$ cm. By the same token, however, the permissible range of values of M_s is sharply reduced for this condition. Thus, for $Y = 0$ (Fig. 24), there is no overlap region for $p_1 = 15$ cm, while meaningful data can be expected for transmission when $6.5 < M_s < 7.1$ and for reflection when $7.1 < M_s < 7.7$. For $p_1 = 1$ cm, the separate Mach number ranges are sharply reduced while an overlap region does exist for $7.3 < M_s < 7.5$. The same qualitative trends are found when a magnetic field is applied. Some overlap region appears when $p_1 = 15$ cm, for $Y = 0.5$ (Fig. 25), while transmission and reflection data can be obtained for the range $6.4 < M_s < 7.2$ and $7 < M_s < 7.8$, respectively. Both intervals are reduced once again at $p_1 = 1$ cm, and a somewhat broader overlap region of $7.2 < M_s < 7.6$ is obtained. Thus, the magnetic field tends to increase the acceptable number of shock-tube conditions in this case.

Having examined the required shock tube conditions on this basis, let us include the corresponding detailed propagation calculations. In Figs. 26 through 30, ϵ_T and ϵ_R are plotted versus Y (or B_0) for these cases. In the present discussions we will assume, somewhat arbitrarily, a maximum practical value of 10,000 gauss. For $p_1 = 15$ cm, useful experimental results may be expected, with respect to transmission (Fig. 26), up to a Mach number of 7.3 (an approximate limit of $X < 2$ is implied). Larger values of M_s, and therefore X, can be examined from the reflection data (Fig. 27). The so-called reversal effect, when an increase in magnetic field results in a decrease in ϵ_T and an increase in ϵ_R, is quite apparent, and experimental information may be obtainable. A qualitatively similar situation is shown in Figs. 28 and 29 for $p_1 = 5$ cm, while several overlap conditions are illustrated in Fig. 30 with $p_1 = 1$ cm.

Fixing $L/\lambda_0 = 4$ once again, for convenience, we shall consider briefly
$f = 15.8$ kMc. The propagation results remain the same since L has been
correspondingly increased; however, the decrease in f raises the shock-tube
performance curves somewhat. The acceptable Mach number range is
generally broadened, as a result, as is seen by comparing Figs. 31 and 32
with 24 and 25, respectively. In addition, the assumed maximum value of
10,000 gauss now corresponds to a larger value of Y. This is shown in Fig.
33, for illustrative purposes, which is to be compared with Fig. 26. With
$M_s = 7.2$ a larger value of X is now involved although the corresponding
variation of ϵ_T with B_0 is more pronounced. The combination $L/\lambda_0 = 2$,
$f = 7.9$ kMc is examined in Figs. 34 through 37. It is immediately apparent
that a larger number of meaningful experimental conditions are now possible,
as are higher values of X and Y. Thus, for $p_1 = 5$ cm (Fig. 34, as compared
with Fig. 28) more values of M_s can be used and a larger value of X exam-
ined. The obvious improvement provided by this condition is shown further
in Fig. 36 for $p_1 = 1$ cm.

Preliminary consideration has been given to a possible shock-tube
program. The resultant analysis clearly shows the need for such calculations
and allows us to obtain meaningful experimental conditions. The plasma
properties, for argon (Figs. 22 and 23), and therefore the corresponding
propagation results, are probably too sensitive to M_s for experimental pur-
poses. A qualitative indication of the problem may be derived from the
present discussion, and the same procedure will be used in a more definitive
analysis of an experimental program using other gases.

SECTION VI

PRELIMINARY ANALYSIS OF THE INHOMOGENEOUS PLASMA PROBLEM

We will be concerned in this section with the problem of examining the more realistic inhomogeneous plasma model. The analyses are preliminary in character in that only typical quantitative effects are obtained from the general considerations. It is expected that this problem will be examined further in some detail in the present research program.

Consider the propagation problem governed by Eq. (52) with the electron density distribution:

$$X(z) = X_{\infty} + (X_0 - X_{\infty}) e^{-az} \quad , \quad z > 0 \quad (a > 0)$$
$$= X_0 \quad , \quad z < 0 \quad . \tag{67}$$

Although the exponential profile has been examined, especially in ionospheric work, as was noted in the summary of Section III, the above form is more suitable for the present purpose. Assuming the incident wave to be linearly polarized along the x-axis gives the solution in the region $z < 0$ as:

$$w = E_y = c^{(1)} \exp[-in_0 (n_0^{*2} - S^2)^{1/2} z] + c^{(2)} \exp[in_0 (n_0^{*2} - S^2)^{1/2} z] \quad , \tag{68}$$

where $n_0^{*2} = 1 - mX_0$ is the constant index of refraction (not necessarily free space). In the region $z > 0$:

$$w'' + n_0^2 \{C^2 - m[X_{\infty} + (X_0 - X_{\infty}) e^{-az}]\} w = 0 \quad . \tag{69}$$

Introducing the substitution:

$$\xi = \frac{2in_0}{a} m^{1/2} (X_0 - X_\infty)^{1/2} e^{-az/2} \quad , \tag{70}$$

we obtain Bessel's equation:

$$\frac{d^2 w}{d\xi^2} + \frac{1}{\xi} \frac{dw}{d\xi} + (1 - \frac{a^2}{\xi^2})w = 0 \quad , \qquad a = \frac{2in_0}{a} (C^2 - mX_\infty)^{1/2} \quad , \tag{71}$$

and the solution:

$$w(\xi) = c^{(3)} J_a(\xi) + c^{(4)} J_{-a}(\xi) \quad . \tag{72}$$

For sufficiently large z, $\xi \ll 1$, and:

$$J_{\pm a}(\xi) \sim \xi^{\pm a} \sim e^{\pm a(-az/2)}$$

The imaginary part of $(C^2 - mX_\infty)^{1/2}$ is negative, such that the real part of a is positive; then, since w must be finite as $z \to \infty$, it follows that the upper sign must be chosen, i.e., $c^{(4)} = 0$:

$$w(\xi) = c^{(3)} J_a(\xi) \quad . \tag{73}$$

At $z = 0$, w and dw/dz must be continuous, and we obtain:

$$R = \frac{c^{(2)}}{c^{(1)}} = \frac{(n_0^{*2} - S^2)^{1/2} - m^{1/2} (X_0 - X_\infty)^{1/2} J_a'(\zeta)/J_a(\zeta)}{(n_0^{*2} - S^2)^{1/2} + m^{1/2} (X_0 - X_\infty)^{1/2} J_a'(\zeta)/J_a(\zeta)} \quad , \quad (74)$$

$$T(\xi) = \frac{c^{(3)}}{c^{(1)}} J_a(\xi) = \frac{2(n_0^{*2} - S^2)^{1/2} J_a(\xi)/J_a(\zeta)}{(n_0^{*2} - S^2)^{1/2} + m^{1/2} (X_0 - X_\infty)^{1/2} J_a'(\zeta)/J_a(\zeta)} \quad , \quad (75)$$

$$T|_{z=0} = T_0 = \frac{2(n_0^{*2} - S^2)^{1/2}}{(n_0^{*2} - S^2)^{1/2} + m^{1/2} (X_0 - X_\infty)^{1/2} J_a'(\zeta)/J_a(\zeta)} \quad , \quad (76)$$

$$\zeta = \frac{2in_0}{a} m^{1/2} (X_0 - X_\infty)^{1/2} \quad . \quad (77)$$

Since Bessel functions of complex order and argument are not tabulated, preliminary quantitative information can best be obtained by an examination of the limiting forms of the solution. Using the series expansion for the Bessel function, we obtain, for:

$$|\zeta| << | \quad , \quad (78)$$

$$R = \bar{R} \left\{ \frac{1 - (in_0/a) \left[(n_0^{*2} - S^2)^{1/2} - (n_\infty^{*2} - S^2)^{1/2} \right]}{1 + (in_0/a) \left[(n_0^{*2} - S^2)^{1/2} + (n_\infty^{*2} - S^2)^{1/2} \right]} \right\} \quad , \quad (79)$$

$$T_0 = \overline{T}_0 \left\{ \frac{1 + (2in_0/a)\,(n_\infty^{*2} - S^2)^{1/2}}{1 + (in_0/a)\left[(n_0^{*2} - S^2)^{1/2} + (n_\infty^{*2} - S^2)^{1/2}\right]} \right\} \quad , \quad (80)$$

where $\overline{R}$ and $\overline{T}_0$ are the limiting values for an abrupt interface separating two homogeneous media:

$$\overline{R} = \frac{(n_0^{*2} - S^2)^{1/2} - (n_\infty^{*2} - S^2)^{1/2}}{(n_0^{*2} - S^2)^{1/2} + (n_\infty^{*2} - S^2)^{1/2}} \quad , \quad (81)$$

$$\overline{T}_0 = \frac{2(n_0^{*2} - S^2)^{1/2}}{(n_0^{*2} - S^2)^{1/2} + (n_\infty^{*2} - S^2)^{1/2}} \quad . \quad (82)$$

Thus, for $|X_0 - X_\infty|$ sufficiently small, or $\underline{a}$, λ_0, Z, or Y sufficiently large, so that $|\zeta| << |$, the reflection coefficient is less than the corresponding well-known result (81) as shown in Eq. (79).

From the asymptotic expansion for the Bessel function or the WKB analysis, it follows that (assuming $\theta_I = 0$ now, for convenience):

$$R = \frac{-a\,(n_\infty^{*2} - n_0^{*2})}{8in_0 n_0^{*3} + a\,(n_\infty^{*2} - n_0^{*2})} \quad , \quad (83)$$

$$T_0 = \frac{8in_0 n_0^{*3}}{8in_0 n_0^{*3} + a\,(n_\infty^{*2} - n_0^{*2})} \quad . \quad (84)$$

-48-

Although it is not necessary to be so restrictive, it will suffice for the present purpose to examine the following limiting expressions. For a rapidly varying exponential profile $(a \to \infty)$ Eqs. (79) and (80) reduce to (for $\theta_I = 0$):

$$R = \overline{R}\left(1 - \frac{2in_0 n_0^*}{a}\right) \quad , \qquad T_0 = \overline{T}_0\left[1 - \left(\frac{in_0}{a}\right)(n_0^* - n_\infty^*)\right] \quad , \quad (85)$$

$$\overline{R} = \frac{(n_0^* - n_\infty^*)}{n_0^* + n_\infty^*} \quad , \qquad \overline{T}_0 = \frac{2n_0^*}{n_0^* + n_\infty^*} \quad . \quad (86)$$

At the other extreme of a slowly varying profile $(a \to 0)$ Eqs. (83) and (84) reduce to:

$$R = \frac{-a(n_\infty^{*2} - n_0^{*2})}{8in_0 n_0^{*3}} \quad , \qquad T_0 = 1 - \frac{a(n_\infty^{*2} - n_0^{*2})}{8in_0 n_0^{*3}} \quad . \quad (87, 88)$$

Rather than undertake a tedious numerical analysis of the formidable exact solutions, (74) and (76), or the simplified expressions, (79), (80) and (83), (84) which would involve the parameters $\underline{a}$, λ_0, X_0, X_∞, Z, and Y, an over-all measure of the influence of the inhomogeneous region can be derived from the limiting relations (85), (86) and (87), (88). These approximate solutions are valid for arbitrary values of n_0^* and n_∞^*, provided that $\underline{a}$ is sufficiently large in (85) and (86) and sufficiently small in (87) and (88). Hence, it is clear that in the present inhomogeneous plasma boundary value problem, nearly zero reflection and perfect transmission is obtained when the electron density is slowly varying. For the rapidly varying case, the reflection and transmission coefficients approach the abrupt interface homogeneous results which may assume essentially arbitrary values, depending on the quantities n_0^* and n_∞^*.

-49-

Additional information can be derived from the following modified version of the preceding boundary value problem which is particularly appropriate in several laboratory applications. If we replace (67) by:

$$X(z) = X_{\infty} + (X_0 - X_{\infty}) e^{-az} \quad , \qquad z > 0 \qquad (a > 0)$$
$$= 0 \quad , \qquad z < 0 \quad ,$$
(89)

the corresponding expressions become:

$$R = \frac{C - m^{1/2} (X_0 - X_{\infty})^{1/2} J_a'(\zeta)/J_a(\zeta)}{C + m^{1/2} (X_0 - X_{\infty})^{1/2} J_a'(\zeta)/J_a(\zeta)} \quad ,$$
(90)

$$T(\xi) = \frac{2C J_a(\xi)/J_a(\zeta)}{C + m^{1/2} (X_0 - X_{\infty})^{1/2} J_a'(\zeta)/J_a(\zeta)} \quad ,$$
(91)

$$T_0 = \frac{2C}{C + m^{1/2} (X_0 - X_{\infty})^{1/2} J_a'(\zeta)/J_a(\zeta)} \quad .$$
(92)

For $|\zeta| << |$:

$$R = \overline{R} \left\{ \frac{1 + (in_0/a) \left[(n_{\infty}^{*2} - S^2)^{1/2} (1 + X_0/X_{\infty}) - (1 - X_0/X_{\infty}) C \right]}{1 + (in_0/a) \left[(n_{\infty}^{*2} - S^2)^{1/2} (1 + X_0/X_{\infty}) + (1 - X_0/X_{\infty}) C \right]} \right\}$$
(93)

$$T_0 = \overline{T}_0 \left\{ \frac{1 + (2in_0/a) (n_{\infty}^{*2} - S^2)^{1/2}}{1 + (in_0/a) \left[(n_{\infty}^{*2} - S^2)^{1/2} (1 + X_0/X_{\infty}) + (1 - X_0/X_{\infty}) C \right]} \right\}$$
(94)

-50-

$$\overline{R} = \frac{C - (n_\infty^{*2} - S^2)^{1/2}}{C + (n_\infty^{*2} - S^2)^{1/2}} \qquad , \qquad \overline{T}_0 = \frac{2C}{C + (n_\infty^{*2} - S^2)^{1/2}} \qquad . \qquad (95, \ 96)$$

In the asymptotic or WKB limit (for $\theta_I = 0$):

$$R = \frac{1 - n_0^* - a\,(n_\infty^{*2} - n_0^{*2})/4in_0 n_0^{*2}}{1 + n_0^* + a\,(n_\infty^{*2} - n_0^{*2})/4in_0 n_0^{*2}} \qquad , \qquad (97)$$

$$T_0 = \frac{2}{1 + n_0^* + a\,(n_\infty^{*2} - n_0^{*2})/4in_0 n_0^{*2}} \qquad . \qquad (98)$$

The further requirement that $\underline{a}$ be very large or very small results in the following (for $\theta_I = 0$):

$$R \overset{a\to\infty}{=} \overline{R}\left[1 - \left(\frac{2in_0}{a}\right)\left(1 - \frac{X_0}{X_\infty}\right)\right] \qquad , \qquad T_0 \overset{a\to\infty}{=} \overline{T}_0\left[1 - \frac{in_0}{a}\left(1 - \frac{X_0}{X_\infty}\right)(1 - n_\infty^*)\right] ,$$

$$\overline{R} = \frac{1 - n_\infty^*}{1 + n_\infty^*} \qquad , \qquad \overline{T}_0 = \frac{2}{1 + n_\infty^*} \qquad , \qquad (99, \ 100)$$

$$R \overset{a\to 0}{=} \frac{1 - n_0^*}{1 + n_0^*}\left[1 - \frac{a(n_\infty^{*2} - n_0^{*2})}{2in_0 n_0^{*2}(1 - n_0^{*2})}\right] \qquad , \qquad T_0 \overset{a\to 0}{=} \frac{2}{1 + n_0^*}\left[1 - \frac{a(n_\infty^{*2} - n_0^{*2})}{4in_0 n_0^{*2}(1 + n_0^*)}\right] \qquad .$$

$$(101, 102)$$

Detailed calculations can be made from the exact solution or, in a restrictive but far more convenient manner, from the corresponding limiting expressions. Equations (99) through (102) are valid for arbitrary values of X_0, X_∞, Z, and Y, provided that $\underline{a}$ is sufficiently large in (99, 100) and sufficiently small in (101, 102). Substantially different results obviously can be obtained from the rapidly varying and slowly varying cases. The solution approach $\bar{R}$ and $\bar{T}_0$ in the former, whereas in the latter the same limiting expressions are obtained with n_∞^* replaced by n_0^*.

There are relatively few solutions of the inhomogeneous plasma problem that include detailed calculations of transmission and reflection. The similarity between Eqs. (41) and (50) allows us to make use of one such solution, that of Albini and Jahn.[21] In particular, the numerical results given in this paper ($B_0 = 0$) for the more complicated slab geometry can be applied to the present magnetoactive problem.

For the case of normal incidence (normal applied magnetic field) and a ramp profile, a linear electron density connecting free space with a homogeneous medium is assumed:

$$
\left.
\begin{aligned}
X(z) &= 0 &, \quad z \le 0 &, \\
&= \frac{\beta z}{L} &, \quad 0 \le z \le L &, \\
&= \beta &, \quad z \ge L &.
\end{aligned}
\right\}
\tag{103}
$$

It is apparent from the previous analysis of Eq. (52) that the reflection and transmission coefficients obtained by Albini and Jahn will apply with a magnetic field present for some equivalent set of physical parameters. When the magnetic field is included:

$$
\left.
\begin{aligned}
N^2 &= n^{*2}(L) = 1 - m\beta \\
&= \left[1 - \frac{\beta(1 \mp Y)^{-1}}{1 + Z^2(1 \mp Y)^{-2}} \right] + i\left[\frac{-\beta Z(1 \mp Y)^{-2}}{1 + Z^2(1 \mp Y^{-2})} \right] \quad .
\end{aligned}
\right\}
\tag{104}
$$

-52-

Selecting two values of N, labeled N_r and N_ℓ, for which the distribution of R and T with L/λ_0 is given by Albini and Jahn, we must obtain the unique values of β, Z, and Y which determine the same curves in the magnetoactive case. For $N_\ell = a - ib$ and $N_r = c - id$, it follows from (104) that:

$$a^2 - b^2 = 1 - \frac{\beta(1 + Y)^{-1}}{1 + Z^2(1 + Y)^{-2}} \quad , \qquad 2ab = \frac{\beta Z (1 + Y)^{-2}}{1 + Z^2 (1 + Y)^{-2}} \quad , \quad (105, 106)$$

$$c^2 - d^2 = 1 - \frac{\beta(1 - Y)^{-1}}{1 + Z^2(1 - Y)^{-2}} \quad , \qquad 2cd = \frac{\beta Z (1 - Y)^{-2}}{1 + Z^2 (1 - Y)^{-2}} \quad . \quad (107, 108)$$

Clearly the solution of (105) through (108) for (β, Z, Y) in terms of the specified quantities (a, b; c, d) is not unique since there are four equations in three unknowns. An additional constraint is implied which will result in the requirement that the choice of N_r and N_ℓ is not entirely arbitrary. Eliminating β from (105, 106) and (107, 108) gives rise to two equations in Z and Y from which we obtain:

$$Y = \frac{cd (a^2 - b^2 - 1) - ab (c^2 - d^2 - 1)}{cd (a^2 - b^2 - 1) + ab (c^2 - d^2 - 1)} \quad , \tag{109}$$

$$Z = \frac{-4abcd}{cd (a^2 - b^2 - 1) + ab (c^2 - d^2 - 1)} \quad . \tag{110}$$

In order that the derived value of β obtained from (105, 106) and (107, 108) be unique, it is necessary that:

$$cd \left[(a^2 - b^2 - 1)^2 + 4a^2b^2 \right] = ab \left[(c^2 - d^2 - 1)^2 + 4c^2d^2 \right] \quad , \tag{111}$$

-53-

in which case β is given by the relation,

$$\beta = -2ab \left[\frac{(c^2 - d^2 - 1)^2 + 4c^2 d^2}{cd(a^2 - b^2 - 1) + ab(c^2 - d^2 - 1)} \right] \quad . \tag{112}$$

A specific value of $N_r = c - id$ and $\underline{a}$ (in $N_\ell = a - ib$) used by Albini and Jahn may be selected. The required value of $\underline{b}$ was obtained from (111). Since in their graphs of $|R|$ versus L/λ_0, for example, a family of curves is presented with $\underline{a}$ fixed and $\underline{b}$ varied, the distribution of $|R_\ell|$ versus L/λ_0 for the derived value of N_ℓ was readily interpolated. The variation of $|R_r|$ and $|R_\ell|$ with L/λ_0 is thus obtained, corresponding to the values of Y, Z, and β which are computed from Eqs. (109), (110), and (112), respectively. It is desirable, of course, to obtain the comparable results with no applied magnetic field. Having determined β and Z, we can calculate $N = e - if$, for $Y = 0$, from the equations:

$$e^2 - f^2 = 1 - \frac{\beta}{1 + Z^2} = A_1 \quad , \qquad 2ef = \frac{\beta Z}{1 + Z^2} = A_2 \quad , \tag{113, 114}$$

i. e. ,

$$e = \left[\frac{A_1 + \left(A_1^2 + A_2^2\right)^{1/2}}{2} \right]^{1/2} \quad , \qquad f = \frac{A_2}{2e} \quad . \tag{115}$$

Since neither e nor f is conveniently specified, a double interpolation of the Albini-Jahn results is required. Illustrative results are shown in Fig. 38 for the condition:

$$\beta = X(L) = 0.82 \quad , \qquad Z = 0.67 \quad , \qquad Y = 1.10 \quad . \tag{116}$$

-54-

The limiting solution as $a \to 0$, Eq. (87), of the semi-infinite problem ($n_0^* = 1$) is identical with the aforementioned Albini-Jahn solution for large values of L/λ_0 with the corresponding terms being $X_\infty = \beta$ and $a = 1/L$. In this case (in which small values of $|R|$ are implied), we obtain:

$$|R| = \frac{\beta}{16\pi \, (L/\lambda_0) \, [(1 \mp Y)^2 + Z^2]^{1/2}} \qquad , \qquad (117)$$

or:

$$\frac{|R|}{|R_0|} = \left[\frac{1 + Z^2}{(1 \mp Y)^2 + Z^2}\right]^{1/2} \qquad , \qquad \frac{\epsilon_R}{\epsilon_{R_0}} = \frac{1}{2}\left(\frac{|R_r|^2}{|R_0|^2} + \frac{|R_\ell|^2}{|R_0|^2}\right) \qquad , \quad (118)$$

where R_0 is the reflection coefficient when $Y = 0$. For $Y \ll 1$, $\epsilon_R \approx \epsilon_{R_0}$, as one would expect. If $Y \gg 1$, $|R_r| \approx |R_\ell| \ll |R_0|$ such that $\epsilon_R \ll \epsilon_{R_0}$, unless Z is much larger than Y in which case, $\epsilon_R < \epsilon_{R_0}$. If $Y = 0(1)$, then $|R_\ell| < |R_0|$; whereas $|R_r| \geq |R_0|$ for $0 \leq Y \leq 2$, and $|R_r| < |R_0|$ for $Y > 2$. Hence, if $Y > 2$, then $\epsilon_R < \epsilon_{R_0}$, and if $0 \leq Y \leq 2$, then ϵ_R may be less than or greater than ϵ_{R_0}; i.e., the applied magnetic field actually may increase the reflected power, depending on the value of Z. When $Y = 1$, for example:

$$\frac{\epsilon_R}{\epsilon_{R_0}} = \left(\frac{1 + Z^2}{2}\right)\left(\frac{1}{Z^2} + \frac{1}{4 + Z^2}\right) \qquad ,$$

hence, $\epsilon_R \gg \epsilon_{R_0}$ for $Z \ll 1$. The effect of L/λ_0, as derived above from the Albini-Jahn calculations for the case of a linear ramp, is illustrated in Fig. 38 for the condition (116). The magnitudes of R_r, R_ℓ, and R_0 vary

relative to one another depending on the value of L/λ_0. Although ϵ_R is generally less than ϵ_{R_0}, as expected $\epsilon_R > \epsilon_{R_0}$ by an order of magnitude for $L/\lambda_0 = 0.8$. Some numerical results are summarized in the following table.

L/λ_0	$|R_0|$	$|R_\ell|$	$|R_r|$	ϵ_R	ϵ_{R_0}
0.1	0.205	0.221	0.105	0.030	0.042
0.5	0.065	0.065	0.017	0.0025	0.00421
0.8	0.015	0.040	0.028	0.00119	0.000225

To evaluate the influence of a finite inhomogeneous region, it is necessary to obtain the corresponding calculation of R_r and R_ℓ for a homogeneous plasma slab of thickness L with the same physical parameters. It is not sufficient, for example, to compare the inhomogeneous medium calculation of $|R|$ for a particular value of L/λ_0 with the corresponding abrupt interface result ($L/\lambda_0 = 0$). A principal conclusion given in the Albini-Jahn paper is that reflection and transmission depend strongly upon the width of the transition zone and, to a lesser extent, on the detailed profile of the transition. The curves presented in their report show the specific influence of the transition dimension L for several profiles, as well as the effect of the index of refraction of the homogeneous medium. It is evident from their results, however, that the propagation may be strongly dependent on the plasma properties as well as on L/λ_0. For the magneto-active plasma, therefore, the effect of the inhomogeneous region on propagation may be quite pronounced under certain conditions, even when L is just a fraction of the free space wavelength.

Clearly the limited results obtained in this fashion are not adequate. A more detailed examination of the effect of a finite inhomogeneous region on propagation can be continued as follows: Compute N_r and N_ℓ from (105) through (108) corresponding to assigned values of X, Y, and Z of interest, in the manner outlined by Eqs. (113) through (115). Reflection and

transmission coefficients can be obtained then from the expressions derived by Albini and Jahn for the ramp, kinked ramp, and trapezoidal profiles, including an applied magnetic field. Additional solutions of (50) for other profiles or geometries would have to be obtained anew in the previously described manner. Varying X, Y, Z, and L/λ_0 will serve to cover the appropriate range of plasma and magnetic field characteristics while several electron density profiles can be used for parametric purposes.

SECTION VII

PROPAGATION IN NONUNIFORM, MAGNETOACTIVE PLASMAS

The magnetic field term in the general equations (17) through (20) can be spatially dependent, as was observed at the end of Section II. This is clearly also the case for the basic wave-like equation (22). Only limited consideration has been given to such problems. This is due in part, perhaps, to the fact that a substantial contribution to the field has been made in ionospherie research wherein the magnetic field is essentially uniform. In the re-entry flight application discussed in Section V, for example, the applied magnetic field is expected to be nonuniform. Therefore, it will be necessary to examine the complex problem of propagation in nonuniform magnetoactive plasmas. The additional difficulty arises from the fact that any nonuniformity in the magnetic field is intrinsically two or three dimensional; hence, the governing differential equations are partial rather than ordinary.

We shall restrict our preliminary attention to the two-dimensional case in whieh the applied magnetie field is in the x-z plane and $\partial/\partial y = 0$. Further, it is assumed to be symmetrical about the z-axis sueh that for $x = 0$, $B_{0x} = 0$ and $B_{0z} = B_{0z}(z)$ while the magnetic field lines diverge for $x \neq 0$ as illustrated in Fig. 39. No current is introduced into the stationary plasma as a result of the application of such a steady dc magnetic field. It follows, since $B_{0y} = \partial/\partial y = 0$, that:

$$\frac{\partial B_{0z}}{\partial x} = \frac{\partial B_{0x}}{\partial z} \quad , \qquad \frac{\partial B_{0x}}{\partial x} = -\frac{\partial B_{0z}}{\partial z} \quad , \tag{119}$$

the last equation resulting from the solenoidal character of $\vec{B}_0$. Assuming a power series expansion about $x = 0$, we can show that:

$$
\left.
\begin{aligned}
B_{0x}(x, z) &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{d^{(2n+1)} B_{0z}(0, z)}{dz^{(2n+1)}} \frac{x^{2n+1}}{(2n+1)!} \quad , \\[4mm]
B_{0z}(x, z) &= B_{0z}(0, z) + \sum_{n=1}^{\infty} (-1)^{n} \frac{d^{(2n)} B_{0z}(0, z)}{dz^{(2n)}} \frac{x^{2n}}{(2n)!} \quad .
\end{aligned}
\right\} \tag{120}
$$

If we let:

$$
E_i(x, z) = \sum_{n=0}^{\infty} E_{in}(z) x^n \quad , \qquad H/_i(x, z) = \sum_{n=0}^{\infty} H/_{in}(z) x^n \quad , \tag{121}
$$

where $i = x, y, z$, it follows from Eqs. (17), (18), (21), (120), and (121) that, to zeroth order in x, the ordinary differential equations (48) apply. In this approximation, one can demonstrate, therefore, the effect of a variable $Y_L(z)$. The effect of some curvature in the magnetic field lines due to the component Y_T and, therefore, some dependence on x can be obtained from the first-order terms in the expansions (121), provided that the quantities $dB_{0z}(0, z)/dz$ and $d^2 B_{0z}(0, z)/dz^2$ are of order one, or that the products $[dB_{0z}(0, z)/dz]x$ and $[d^2 B_{0z}(0, z)/dz^2]x^2$ are sufficiently small. In solving for the higher order terms, it will be necessary to appropriately order the plasma properties X and Z.

Although Eqs. (48) are clearly more complicated when Y_L is a function of z than for the case of a variable X previously considered, preliminary

results can be derived based on the above expansion procedure. It is reasonable to assume, for the present purpose, that the applied magnetic field is of the form:

$$Y_L(z) = Y_0 e^{-az} \qquad , \qquad a > 0 \qquad . \tag{122}$$

For $z > 0$:

$$F'' + n_0^2 \left[1 - \frac{X}{U} \left\{ 1 - \frac{[\mp Y_0 \exp(-az)]}{U} \right\} \right] F = 0 \qquad , \qquad |U| >> Y_0 e^{-az} \qquad . \tag{123}$$

Introducing the substitution:

$$\psi = \frac{2n_0}{a} \frac{X^{1/2}}{U} (\mp Y_0)^{1/2} e^{-az/2} \qquad , \tag{124}$$

we obtain the general solution:

$$F = c^{(3)} J_\beta(\psi) + c^{(4)} J_{-\beta}(\psi) \qquad , \qquad \beta = \frac{2in_0}{a} \left(1 - \frac{X}{U} \right)^{1/2} = \frac{2in_0 n_0^*}{a} \qquad , \tag{125}$$

where n_0^* is the index of refraction with zero applied magnetic field. For sufficiently large z, $\psi << 1$, and:

$$J_{\pm\beta}(\psi) \sim \psi^{\pm\beta} \sim e^{(\pm\beta)(-az/2)} \qquad .$$

Since the imaginary part of n_0^* is negative, $\mathrm{Re}\, \beta > 0$, and the upper sign must be chosen in order to satisfy the boundary condition at $z = \infty$:

$$F(\psi) = c^{(3)} J_\beta(\psi) \qquad . \tag{126}$$

-61-

We shall assume the region $z < 0$ to be free space and the incident wave linearly polarized along the x-axis. Hence:

$$F(z) = c^{(1)} e^{-in_0 z} + c^{(2)} e^{in_0 z} \quad , \qquad \text{for } z < 0 \quad . \tag{127}$$

Applying the boundary conditions at $z = 0$, we obtain:

$$R = \frac{c^{(2)}}{c^{(1)}} = \frac{1 + i(X^{1/2}/U)(\mp Y_0)^{1/2} J'_\beta(\phi)/J_\beta(\phi)}{1 - i(X^{1/2}/U)(\mp Y_0)^{1/2} J'_\beta(\phi)/J_\beta(\phi)} \quad , \tag{128}$$

$$T(\psi) = \frac{2J_\beta(\psi)/J_\beta(\phi)}{1 - i(X^{1/2}/U)(\mp Y_0)^{1/2} J'_\beta(\phi)/J_\beta(\phi)} \quad , \tag{129}$$

$$T(z = 0) = T_0 = \frac{2}{1 - i(X^{1/2}/U)(\mp Y_0)^{1/2} J'_\beta(\phi)/J_\beta(\phi)} \quad , \tag{130}$$

$$\phi = \frac{2n_0}{a} \frac{X^{1/2}}{U} (\mp Y_0)^{1/2} \quad . \tag{131}$$

For $|\phi| \ll 1$,

$$R = \left(\frac{1 - n_0^*}{1 + n_0^*}\right) \left\{ \frac{1 - [(in_0/a)(\mp Y_0)(1 + n_0^*)]/U(1 + 2in_0 n_0^*/a)}{1 - [(in_0/a)(\mp Y_0)(1 - n_0^*)]/U(1 + 2in_0 n_0^*/a)} \right\} \quad , \tag{132}$$

$$T_0 = \left(\frac{2}{1 + n_0^*}\right) \left\{ \frac{1}{1 + [(in_0/a)(\mp Y_0)(1 - n_0^*)]/U(1 + 2in_0 n_0^*/a)} \right\} \quad . \tag{133}$$

Since $|Y_0 \exp(-az)| \ll |U|$, it follows that $|\phi| \ll 1$ provided that, $|(2n_0/a)X^{1/2}/(\mp Y_0)^{1/2}|$ is of order one, or smaller.

In the asymptotic or WKB limit:

$$R = \frac{1 - n_c^* + aX(\mp Y_0)/4in_0 U^2 n_c^{*2}}{1 + n_c^* - aX(\mp Y_0)/4in_0 U^2 n_c^{*2}} \qquad , \tag{134}$$

$$T_0 = \frac{2}{1 + n_c^* - aX(\mp Y_0)/4in_0 U^2 n_c^{*2}} \qquad , \tag{135}$$

where the index of refraction:

$$n_c^* = \left(1 - \frac{X}{U \mp Y_0}\right)^{1/2} \approx \left\{1 - \frac{X}{U}\left[1 - \frac{(\mp Y_0)}{U}\right]\right\}^{1/2} \qquad , \qquad |U| >> Y_0 \qquad , \tag{136}$$

characterizes the uniform magnetoactive problem. It follows that:

$$R \overset{a \to \infty}{=} \overline{R}\left[1 - \frac{2in_0 n_0^*(\mp Y_0)}{aU}\right] \qquad , \qquad \overline{R} = \frac{1 - n_0^*}{1 + n_0^*} \qquad , \tag{137}$$

$$T_0 \overset{a \to \infty}{=} \overline{T}_0\left[1 - \frac{in_0 X(\mp Y_0)}{U^2 a(1 + n_0^*)}\right] \qquad , \qquad \overline{T}_0 = \frac{2}{1 + n_0^*} \qquad , \tag{138}$$

$$R \overset{a \to 0}{=} R^*\left[1 + \frac{aX(\mp Y_0)}{2in_0 n_c^{*2} U^2\left(1 - n_c^{*2}\right)}\right] \qquad , \qquad R^* = \frac{1 - n_c^*}{1 + n_c^*} \qquad , \tag{139}$$

$$T_0 \overset{a \to 0}{=} T_0^*\left[1 + \frac{aX(\mp Y_0)}{4in_0 n_c^{*2} U^2(1 + n_c^*)}\right] \qquad , \qquad T_0^* = \frac{2}{1 + n_c^*} \qquad . \tag{140}$$

Specific calculations can be made readily using the limiting expressions (132) through (136), which cover a reasonably broad range of conditions. Equations (132) and (133) are appropriate, for example, if Y_0 is the same order of magnitude as X, and if $\underline{a}$ is the order of $4\pi/\lambda_0$, or larger. It is apparent from the simplified relations (137) through (140), however, that if the field is slowly varying, the solution approaches the constant magnetic field results R* and T_0^*. By contrast, for a rapidly varying field, the solution approaches $\overline{R}$ and $\overline{T}_0$ in which there is no effect of the applied magnetic field. This general observation is a clear indication of the possible substantial importance of a nonuniform, applied, magnetic field. A broad indication of this type is consistent with the present purpose. Further analysis of the initial expansion procedure, including a plasma slab geometry, for example, would be required before a detailed quantitative analysis would be warranted. It appears that an experimental investigation of this problem may be feasible after the initial study described earlier in this report has been completed.

The analysis for a nonuniform, applied, magnetic field is dependent, certainly, on the character of the nonuniformity. In the present discussion, this nonuniformity is due to the application of a magnetic field which is of the type one might reasonably expect to use to improve transmission in actual flight applications. The preliminary consideration of a formal expansion procedure resulted in a general evaluation of the effect of a longitudinal variable magnetic field. A more complete treatment of the partial differential equations is outlined so as to account for the distortion of the wave forms resulting from the dependence of the index of refraction on x. Other forms of a nonuniform, applied, magnetic field may be postulated. It should be remarked also that even when $\vec{Y} \equiv Y_L$ = constant, the interaction of the flow with the applied magnetic field results in an induced nonuniformity in the field as well as inhomogeneities in the medium. In the magnetogasdynamic sense, these perturbations generally are expected to be small. It may be shown readily, however, that small changes in the thermodynamic properties of the gas, and therefore in the plasma properties, or small, induced nonuniformities in the magnetic field may be of significance in several applications of interest.

SECTION VIII

SUMMARY

The basic equations governing the propagation of electromagnetic waves in inhomogeneous, dissipative, nonuniform, magnetoactive plasmas are derived in Section II. A review of the present state of the art for plane harmonic waves is presented in Sections III and IV. The plasma is assumed to be homogeneous in the former, and the essential features of ray theory are discussed along with the limiting condition as prescribed by the WKB approximation. In Section IV, the general inhomogeneous plasma problem is described. Some consideration is given to the magneto-ionic coupling theory for a stratified medium assuming constant collision frequency and an arbitrary orientation of the uniform, applied magnetic field. Particular emphasis is placed, however, on the problem of propagation across an abrupt plane ($z = 0$) interface into an inhomogeneous plasma region where the incident wave may be oblique (its wave normal in the x-z plane) if there is no applied magnetic field but must be normal in the uniform, magnetoactive problem. The governing ordinary differential equations are derived with the magnetic field in the x-z plane; however, detailed consideration is given to the problem in which the field is applied in the direction of propagation, i.e., normal to the interface.

Exact solutions for a number of assumed functional forms of the index of refraction are summarized along with various boundary value problems that have been considered. The utility of these formal solutions is limited by the difficulty encountered in abstracting numerical results. Detailed calculations have been made, therefore, in only a few instances. As is usually the case when analytical expressions are available, however, considerable information can be obtained from the limiting forms of these solutions. Numerical and approximate analytical procedures are discussed

which may be used to obtain solutions for more complicated forms of the index of refraction or in the analysis of more general propagation problems referred to above. The reader is referred to the bibliographies and reviews of Brekhovskikh,[67] Spence,[68] Evans,[69] and Owens,[70] as well as to those already cited in the text, for a more general treatment of the propagation problem as well as additional references.

Mention should be made, at least in the summary, of ionospheric research with regard to the propagation of Whistlers. Whistlers are low-frequency electromagnetic wave packets that propagate along magnetic lines in the ionosphere. They are of possible interest for the present purpose since the appropriate frequency ratios, $X > Y \gg 1$, are of the same order of magnitude as were found in several applications discussed in this report. Although Whistler theory may, therefore, be applied to a certain extent in such cases, it is unfortunately not too far advanced. Current work is primarily concerned with ray propagation. The transmission of wave packets and their group velocity is analyzed. This is used in the study of transmission of momentary disturbances such as electromagnetic radiation from lightning bolts in the atmosphere, the source of atmospheric Whistlers. Use of ray theory in regimes of large Y, where Whistlers propagate, is generally found to be valid when the medium properties, and therefore the electromagnetic fields, are slowly varying. Although full wave solutions have been found for propagation at oblique angles to uniform magnetic fields, the manner in which the electromagnetic waves propagate along the magnetic lines is not always clear. Reference has been made to the paper of French, Cloutier, and Bachynski,[54] in which the existence of very low-frequency "windows" due to the ion cyclotron mode was pointed out. This also has been observed in the study of Whistlers and may be of definite interest in the present study. Additional detail is given, for example, by Budden,[4] Hines,[71] Gallet,[72] and Hoffman.[73]

The present preliminary analysis of the effect of an applied magnetic field on propagation is contained in Sections V through VII. The initial quantitative treatment of the problem was made in Section V using the

simplified model of a homogeneous plasma slab, assuming normal incidence
and normal applied, uniform, magnetic field. A detailed parametric calcu-
lation of transmission and reflection was made for a wide range of values of
X, Z, Y, and L/λ_0. Illustrative transmission results are given in Figs. 3
through 16. It is apparent immediately that the manner in which the mag-
netic field affects transmission is dependent upon a number of significant
parameters. The size of the field is obviously a factor, but so, for example,
are the values of db_T, X, Z, L/λ_0, and f involved. The combined effect of
all of the parameters is discussed in detail in Subsection A.

It is clear, however, that substantial improvement in transmission
generally can be derived from the application of reasonable field strengths
for moderately overdense plasmas. This is also the case when $X \gg 1$ pro-
vided that one accepts as much as 10 to 20 db degradation in the transmitted
signal, in contrast to the requirement of, at times, an impractical order of
magnitude larger field when near-complete transmission is considered.
Several exemplary re-entry situations were analyzed in the second subsection.
The problem of a 10-degree half-angle cone is summarized in Fig. 17, and
corresponding transmission results are given in Figs. 18 through 21. The
effect of the magnetic field in eliminating blackout is discussed for the flight
conditions and signal frequencies considered assuming several different
levels of acceptable transmission. It should be noted that a preliminary
analysis of the boundary layer on a 10-degree cone clearly indicates the
existence of electron density gradients in the direction of propagation. The
magnetic field required to provide a specified acceptable level of transmission
can be reduced substantially when the more realistic inhomogeneous plasma
is considered, as is demonstrated in Section VI. An experimental shock-tube
program is briefly analyzed at the conclusion of this section. Although the
resultant range of conditions (f_p and f_c) which may thus be obtained using
argon is satisfactory, their dependence on the shock Mach number appears
to be too strong for acceptable experimental work. A qualitative appreciation
of the problem may be obtained from this discussion, however, and the pro-
cedure will be repeated using other gases in subsequent detailed studies of

-67-

the program. It is at once clear from the theoretical calculations that an arbitrary choice of shock-tube conditions could lead easily to the collection of virtually meaningless experimental results. Using the solution for ϵ_T and ϵ_R with the constraints that these quantities and their derivatives with respect to X be within prescribed bounds, we have found that the possible experimental conditions (M_s, p_1) are well delineated; this is vividly demonstrated in Figs. 24, 25, 31, and 32. The variation of ϵ_T and ϵ_R with B_0 for these conditions is shown in Figs. 26 through 30 and 33 through 37.

It was noted above that the variation of db_T with Y becomes more gradual for lower values of db_T, particularly when $X \gg 1$. It follows that such effects as plasma inhomogeneities and nonuniformities in the magnetic field, even if they are relatively small, can be of considerable practical importance. A less accurate theoretical calculation of db_T may result in a significant error in the prediction of the applied magnetic field requirement which is subject to definite practical limitations. In Fig. 4b, for example, for $X = 100$, a 25-percent decrease in the value of db_T (for 10 db or less) would result in at least a 50-percent increase in one's estimate of the field strength. The use of a homogeneous plasma slab model for the analysis of a re-entry cone could easily produce such errors in the prediction of db_T. Hence, the more realistic inhomogeneous plasma slab model is considered in Section VI. An exponential distribution of electron density is assumed. The plasma is taken to be semi-infinite in extent, a simplifying assumption that is appropriate in certain laboratory applications but is considerably more approximate in the flight applications of present interest. Exact solutions to two boundary value problems are obtained involving Bessel functions of the first kind with complex order and argument. Considerably simplified expressions are obtained from the small and large argument expansions of the solutions. The over-all effect of the inhomogeneity is immediately evident, however, since the limiting expression for a large electron density gradient approaches the well-known homogeneous result, whereas substantially different values may be obtained from the corresponding case in which the gradients are small. Quantitative results of the effect of an

-68-

applied magnetic field on the reflected power for the inhomogeneous plasma problem are obtained using the solution (87), i.e., for a $\to 0$ (taking $n_0^* = 1$). This is identical to the solution for a linear ramp in the limit of large L/λ_0. The relative size of $|R_r|$, $|R_\ell|$, and $|R_0|$ will vary depending on the values of Z and Y. Although ϵ_R is generally less than ϵ_{R_0}, as expected, the applied magnetic field actually can increase the reflected power. Indeed, it is found that for Y = 1, $\epsilon_R \gg \epsilon_{R_0}$ when Z << 1. The effect of L/λ_0 is derived from the Albini-Jahn calculations for the case of a linear ramp and is illustrated in Fig. 38 for one set of values of β, Z, and Y. The reflected energy depends on L/λ_0, and although it is generally smaller in the magnetoactive case, it is interesting to note that for $L/\lambda = 0.8$, $\epsilon_R > \epsilon_{R_0}$ by an order of magnitude.

A preliminary analysis of propagation in a semi-infinite, homogeneous, nonuniform, magnetoactive plasma is made in Section VII. The incident wave is normal to the free space-plasma interface (z = 0), and the magnetic field is applied in the x-z plane, symmetrical about the z-axis, as shown in Fig. 39, where $B_{0x}(0, z) = 0$ and $B_{0z}(0, z) = B_{0z}(z)$. Such a geometry is a reasonable approximation to what one might expect in actual flight applications of the use of magnetic fields to improve transmission. A formal expansion procedure in powers of x is proposed which, to order zero, reduces to the usual ordinary differential equation (50) in which the magnetic field has only a variable longitudinal component. Assuming that $Y_L(z) = Y_0 e^{-az}(a > 0)$, $Y_L << |U|$, this boundary value problem is solved in terms of Bessel functions of the first kind of complex order and argument. The limiting forms of this solution for small and large values of the argument are also derived. A general observation of the possible significance of the nonuniformity can be determined from the expression for the case of a slowly varying and rapidly varying field strength. In the former limit, the solution approaches the corresponding constant magnetic field result, while in the latter case the favorable effect of the magnetic field is essentially lost. The fact that applied magnetic fields will not be uniform in such applications, along with the fact that the magnetogasdynamic effect will induce nonuniformities in the field (as well as plasma inhomogeneities), requires that this complicated problem be given further consideration.

-69-

SECTION IX

FIGURES
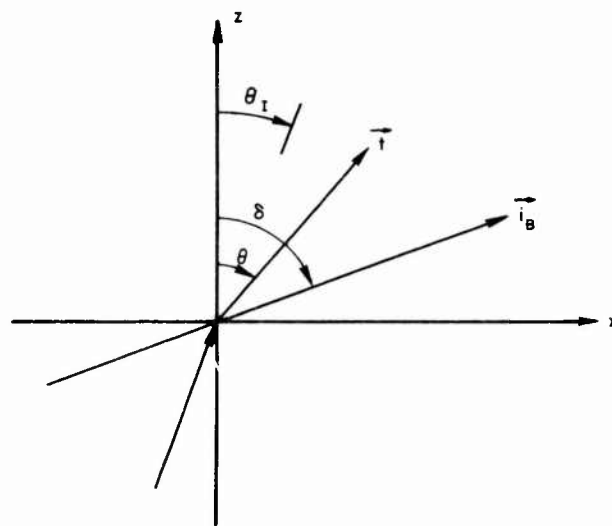
Fig. 1. Propagation geometry
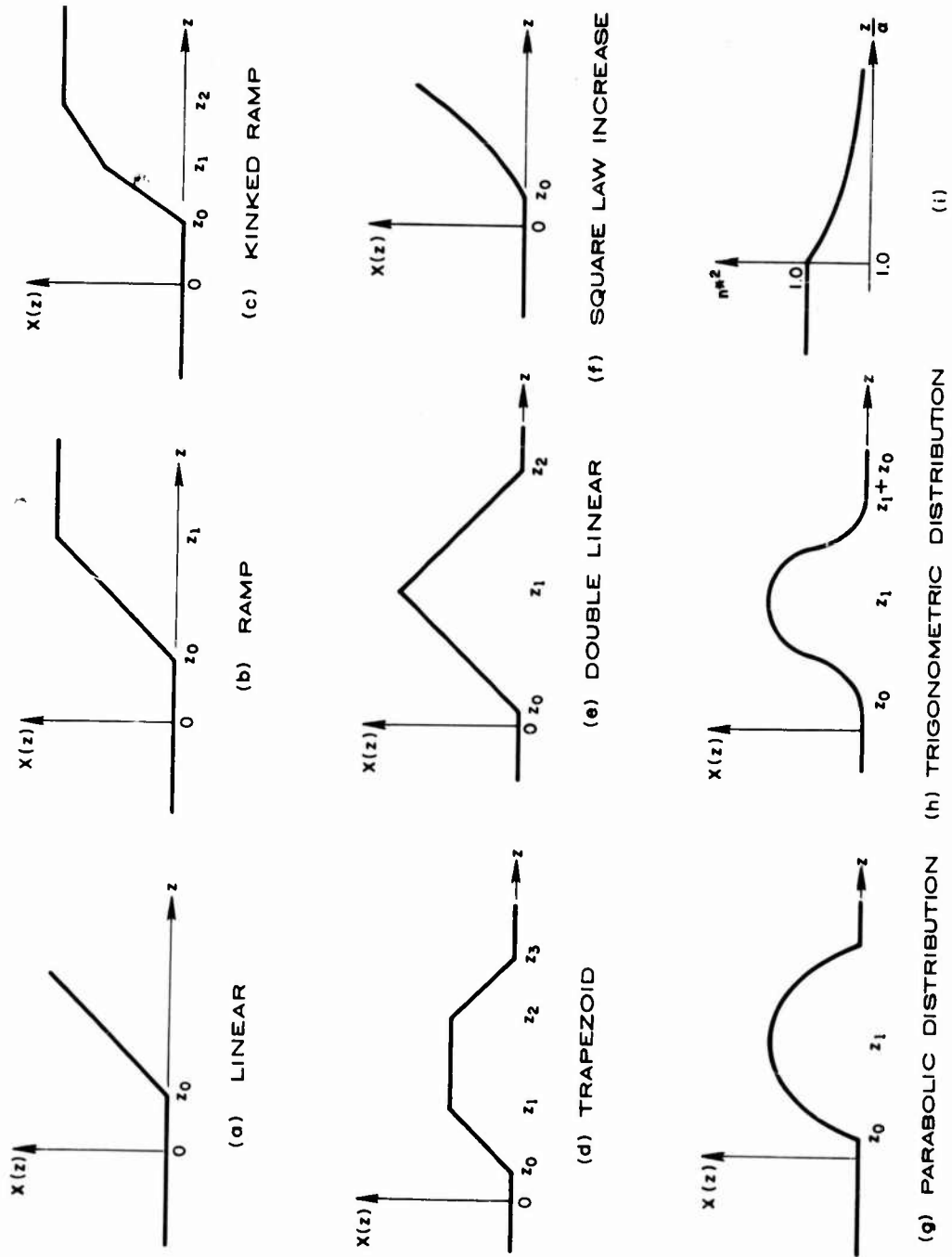
Fig. 2. Inhomogeneous plasma boundary value problems [see summary following Eq. (52)]

(a) LINEAR

(b) RAMP

(c) KINKED RAMP

(d) TRAPEZOID

(e) DOUBLE LINEAR

(f) SQUARE LAW INCREASE

(g) PARABOLIC DISTRIBUTION

(h) TRIGONOMETRIC DISTRIBUTION

(i)

-72-
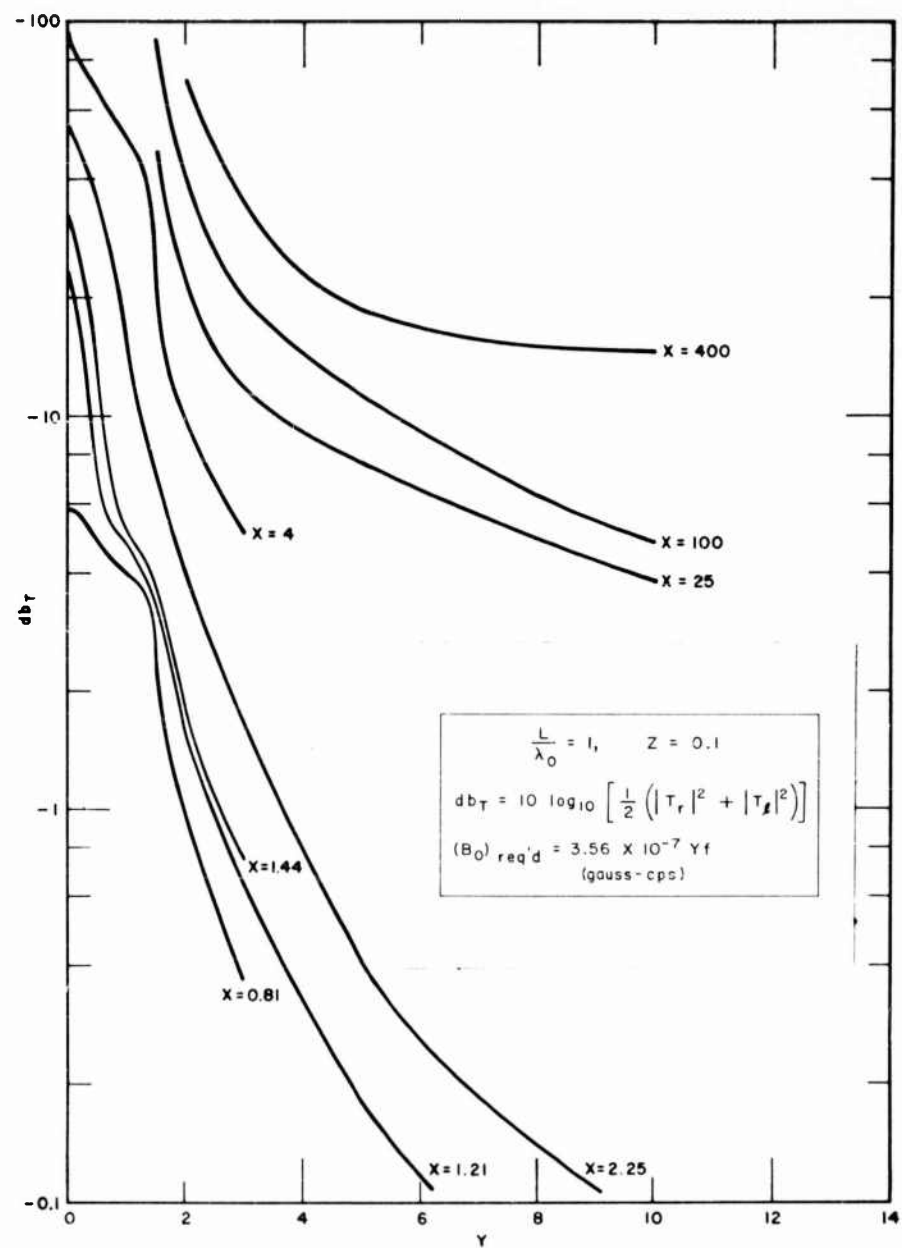
Fig. 3a. db_T versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 0.1$

-73-
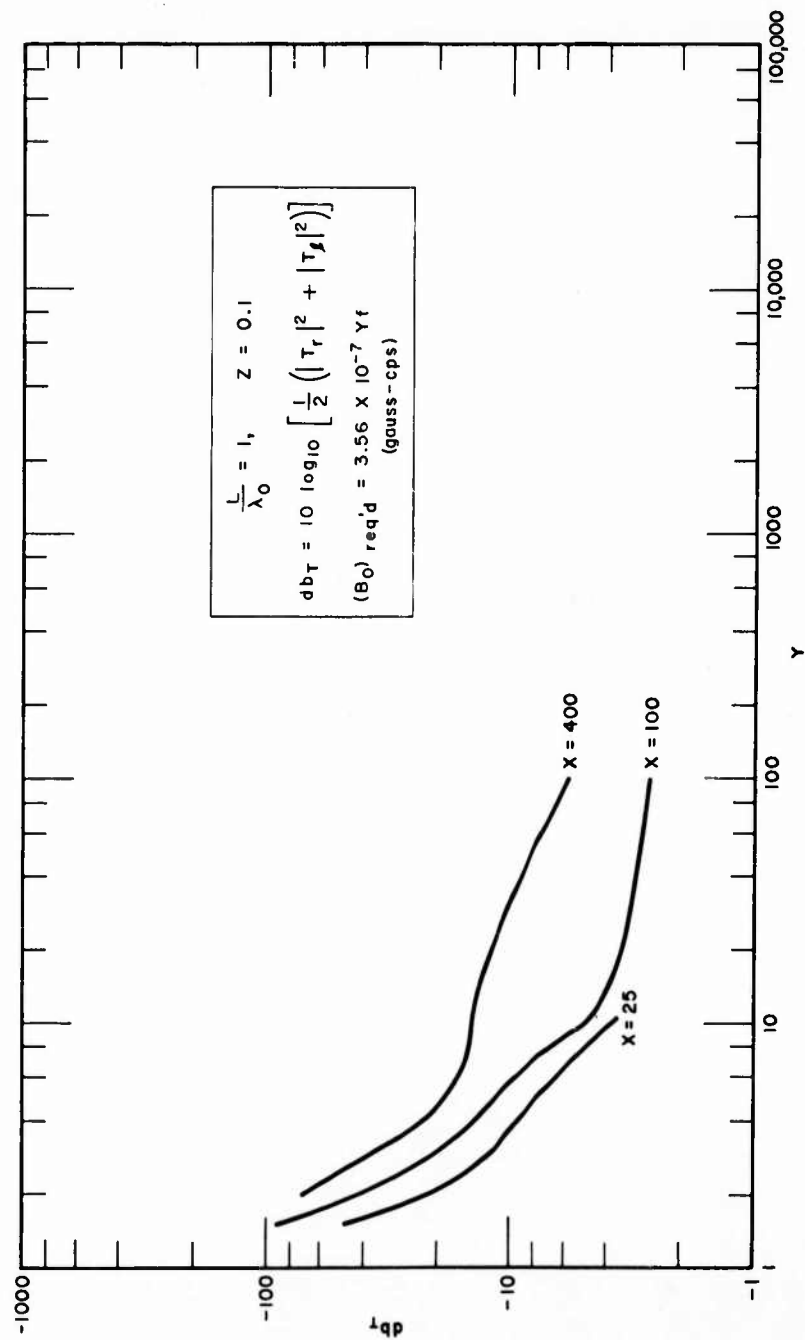
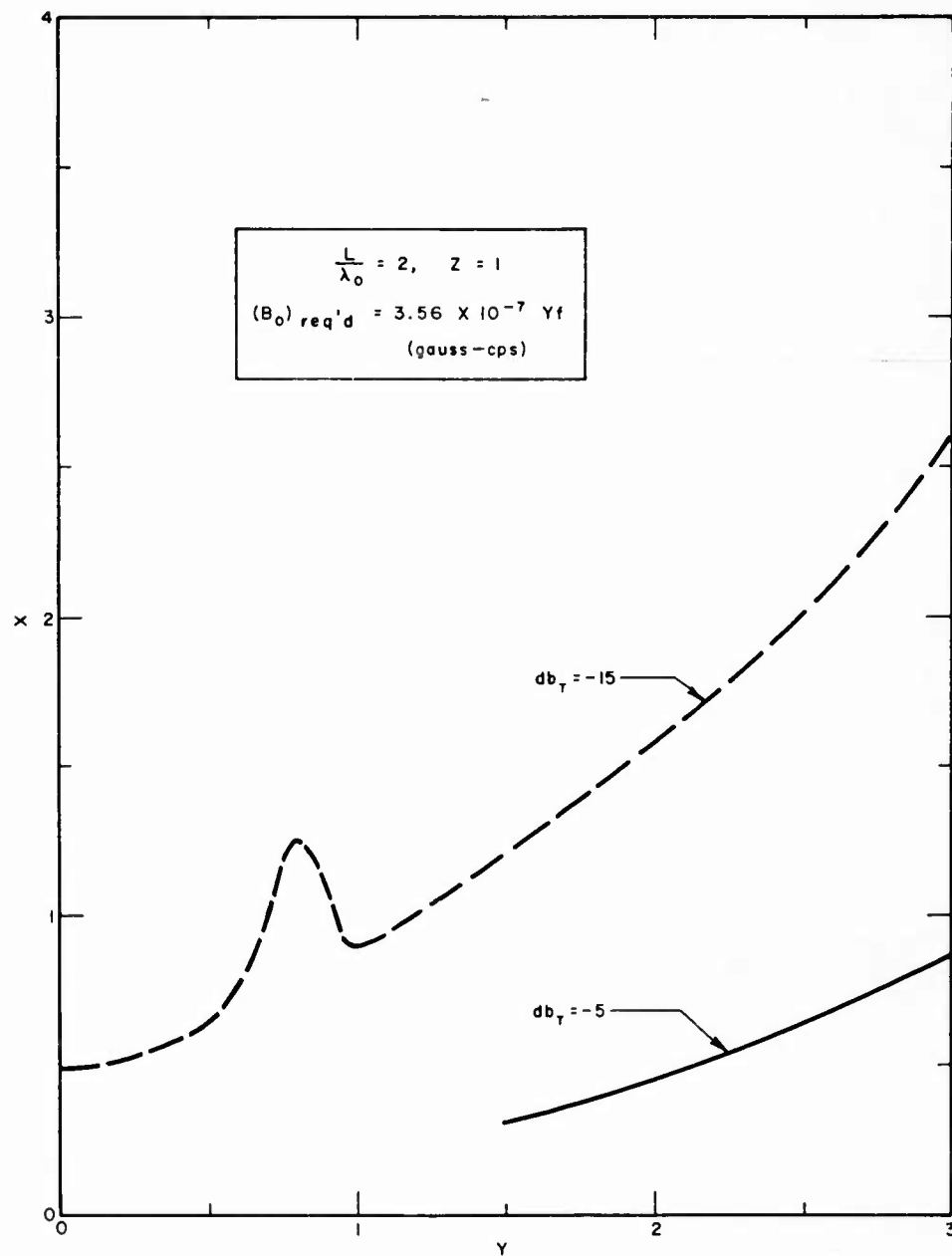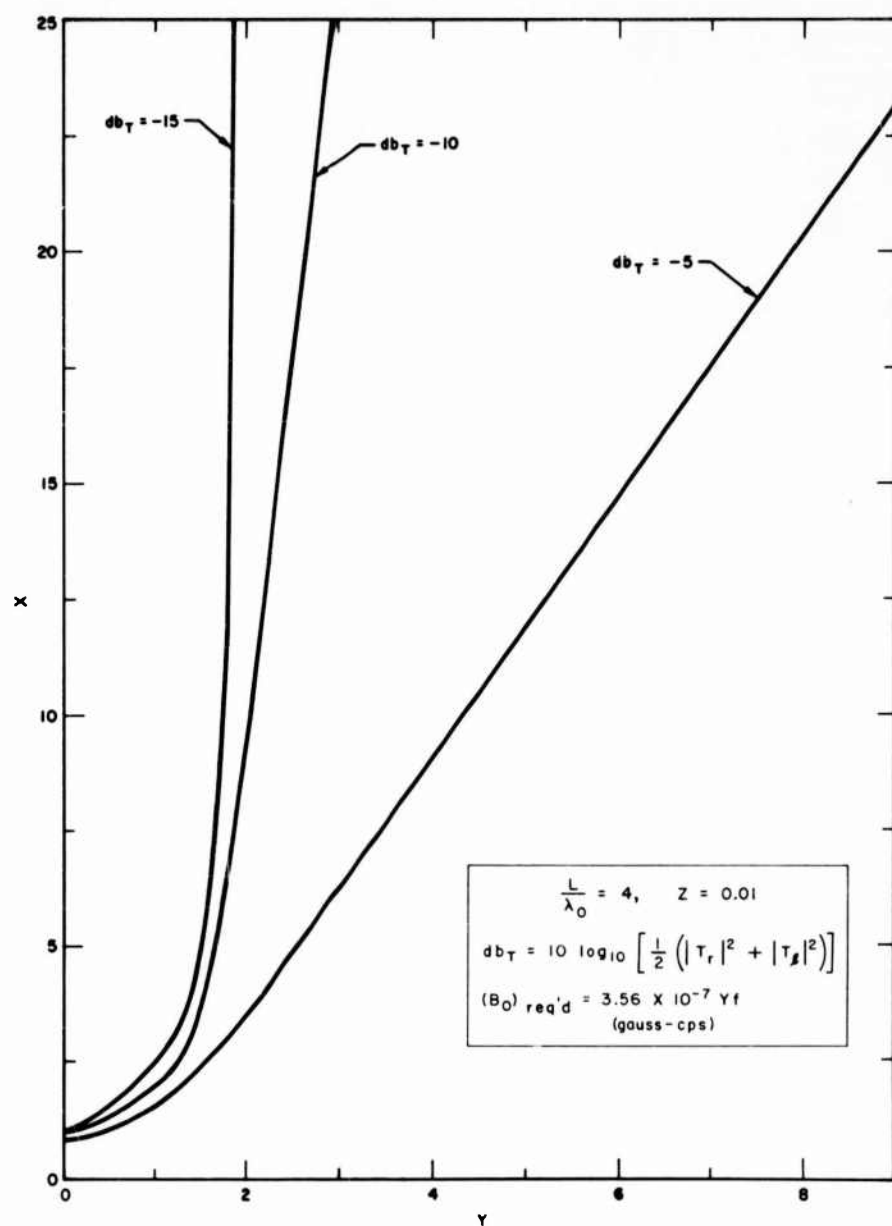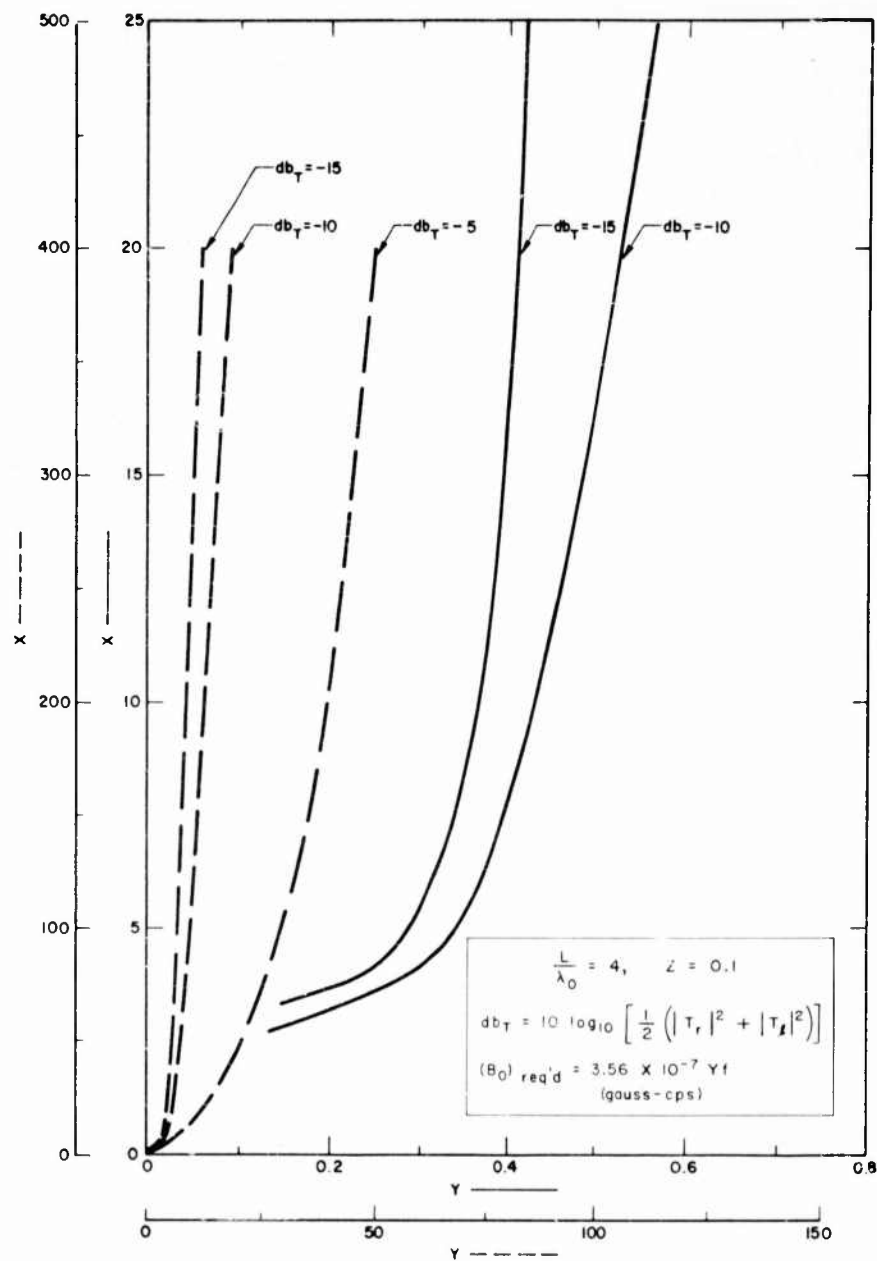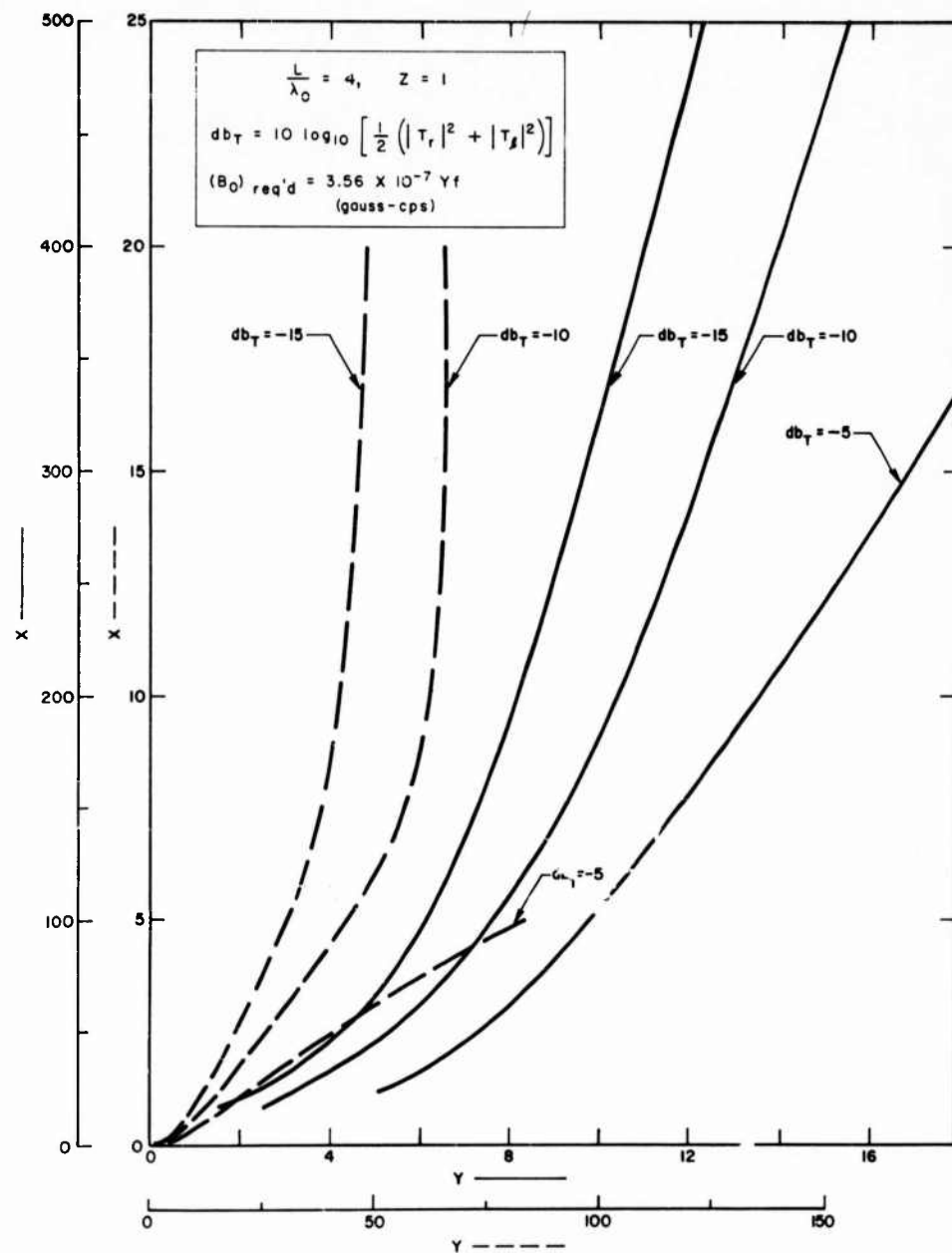Fig. 3b. db_T versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 0.1$
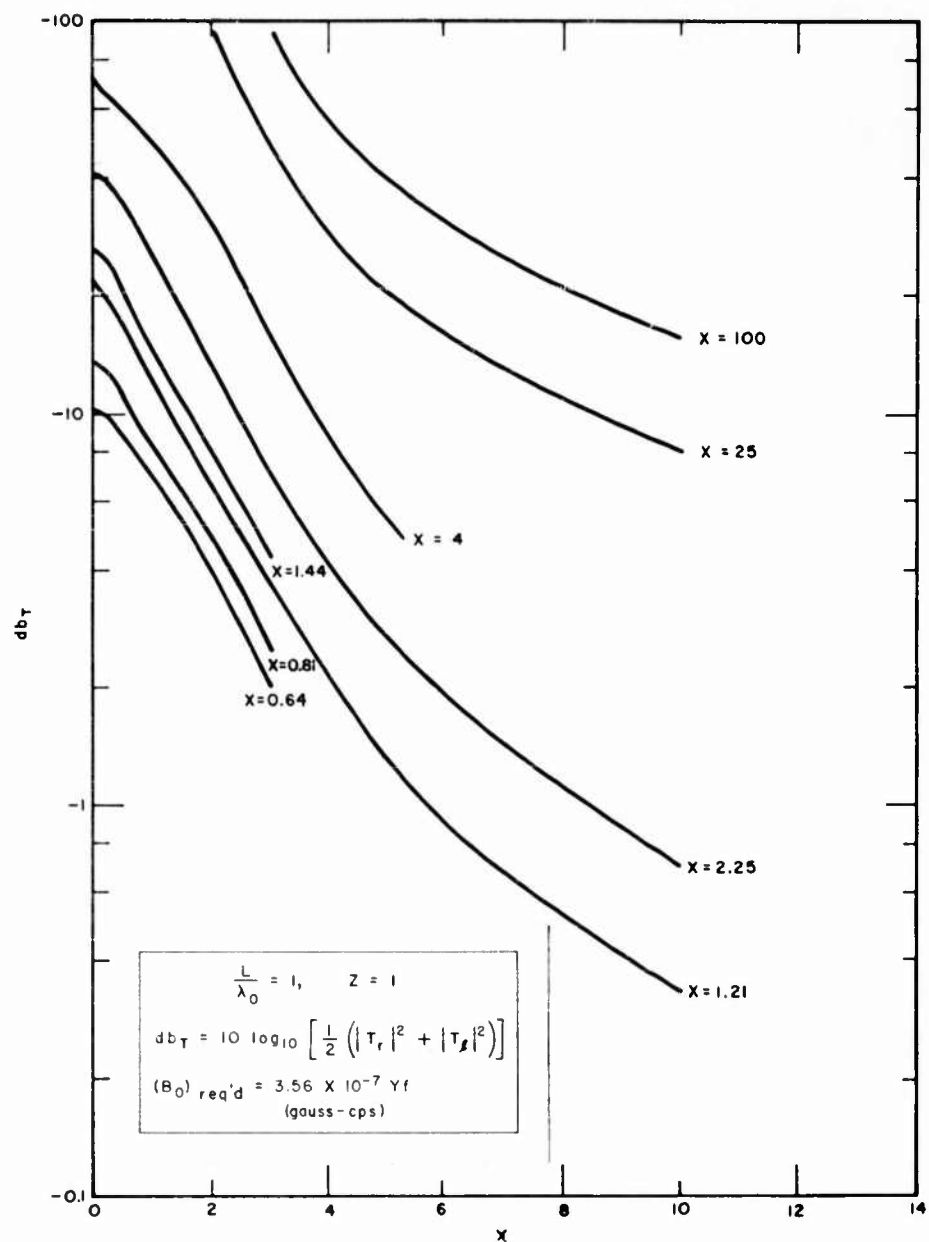
Fig. 4a. db_T versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 1$

Fig. 4b. db_T versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 1$

The figure contains the following annotations:

$$\frac{L}{\lambda_0} = 1, \quad Z = 1$$

$$db_T = 10 \log_{10} \left[\frac{1}{2} \left(|T_r|^2 + |T_\ell|^2 \right) \right]$$

$$(B_0) \text{ req'd} = 3.56 \times 10^{-7} \, Y_f$$

(gauss-cps)

Curves labeled: X = 400, X = 100, X = 25
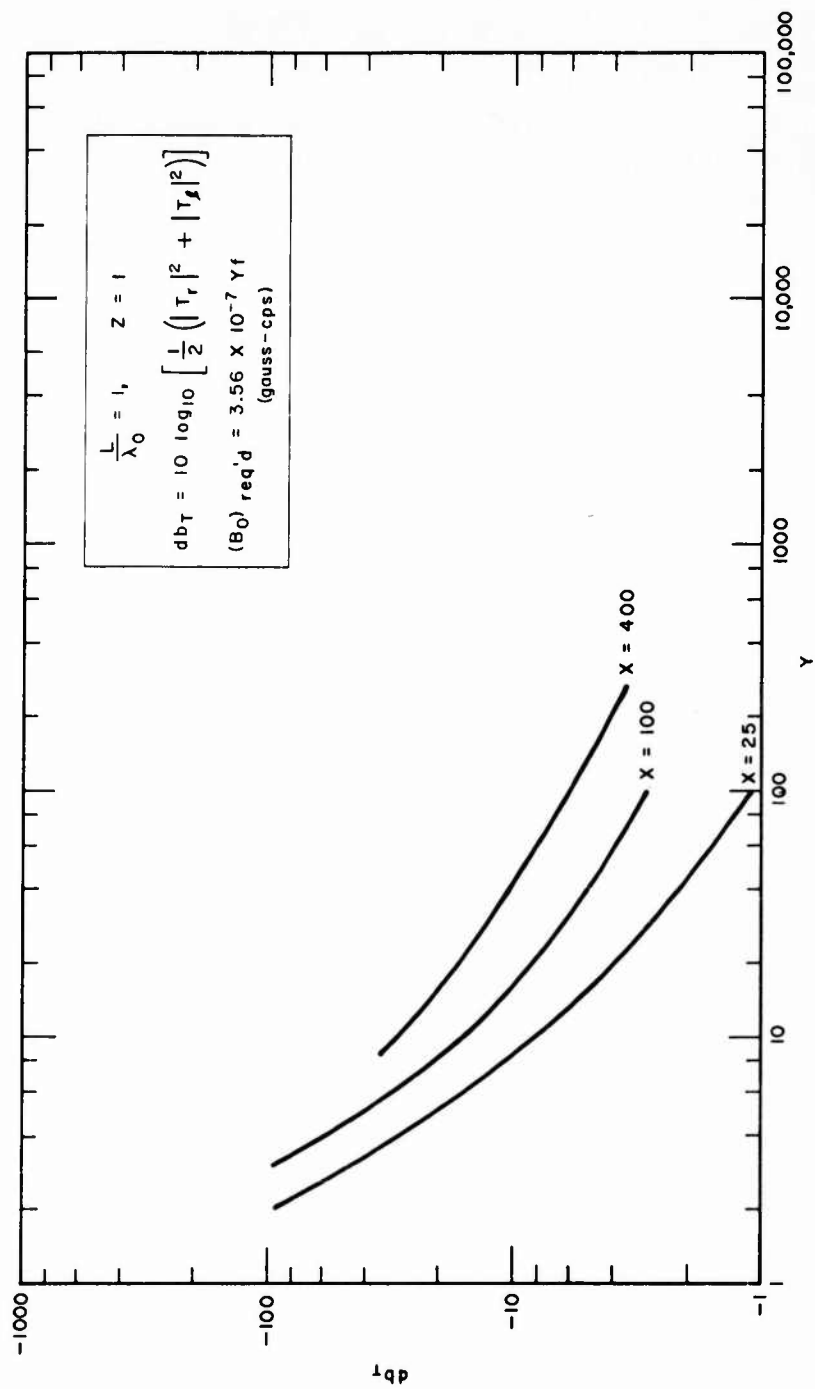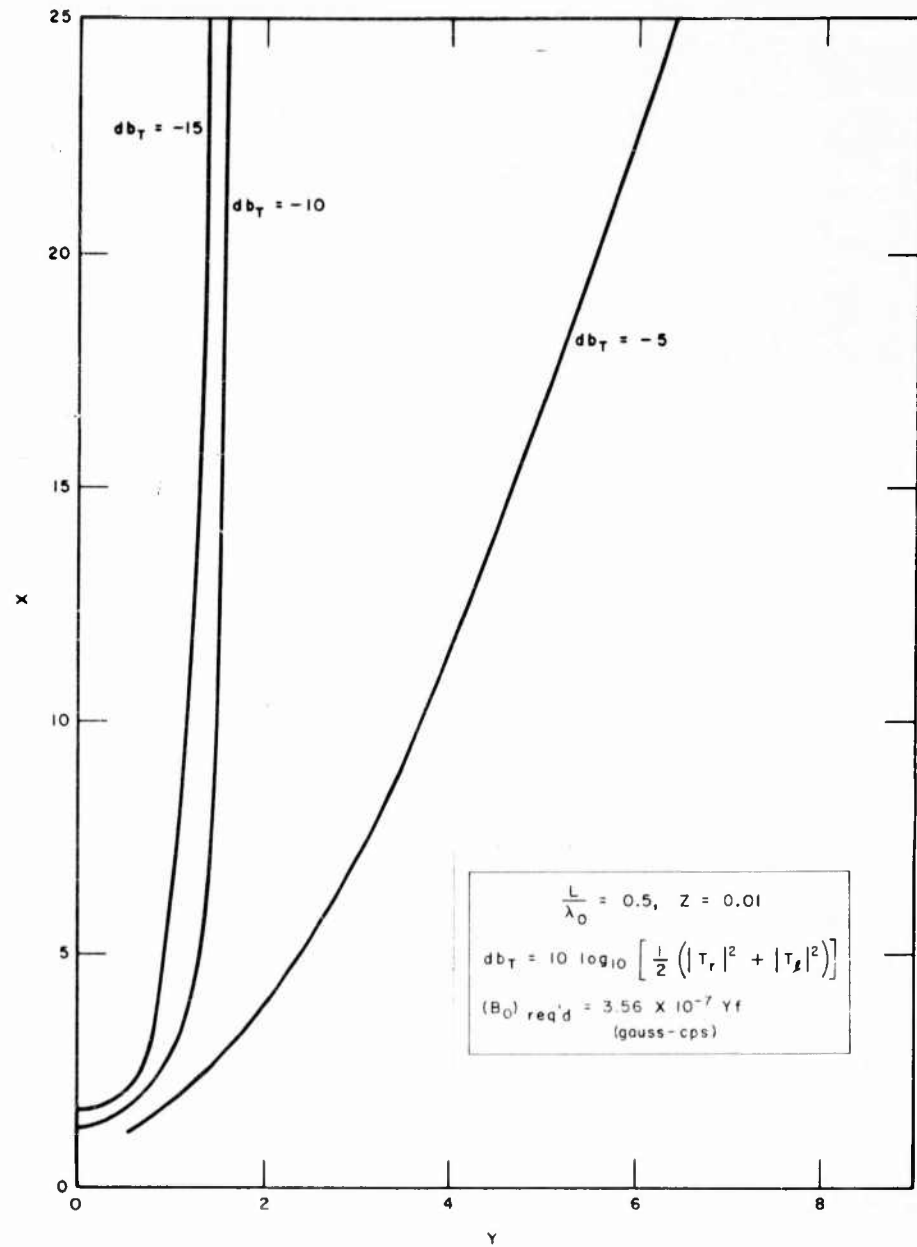
Fig. 5. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 0.5$, $Z = 0.01$
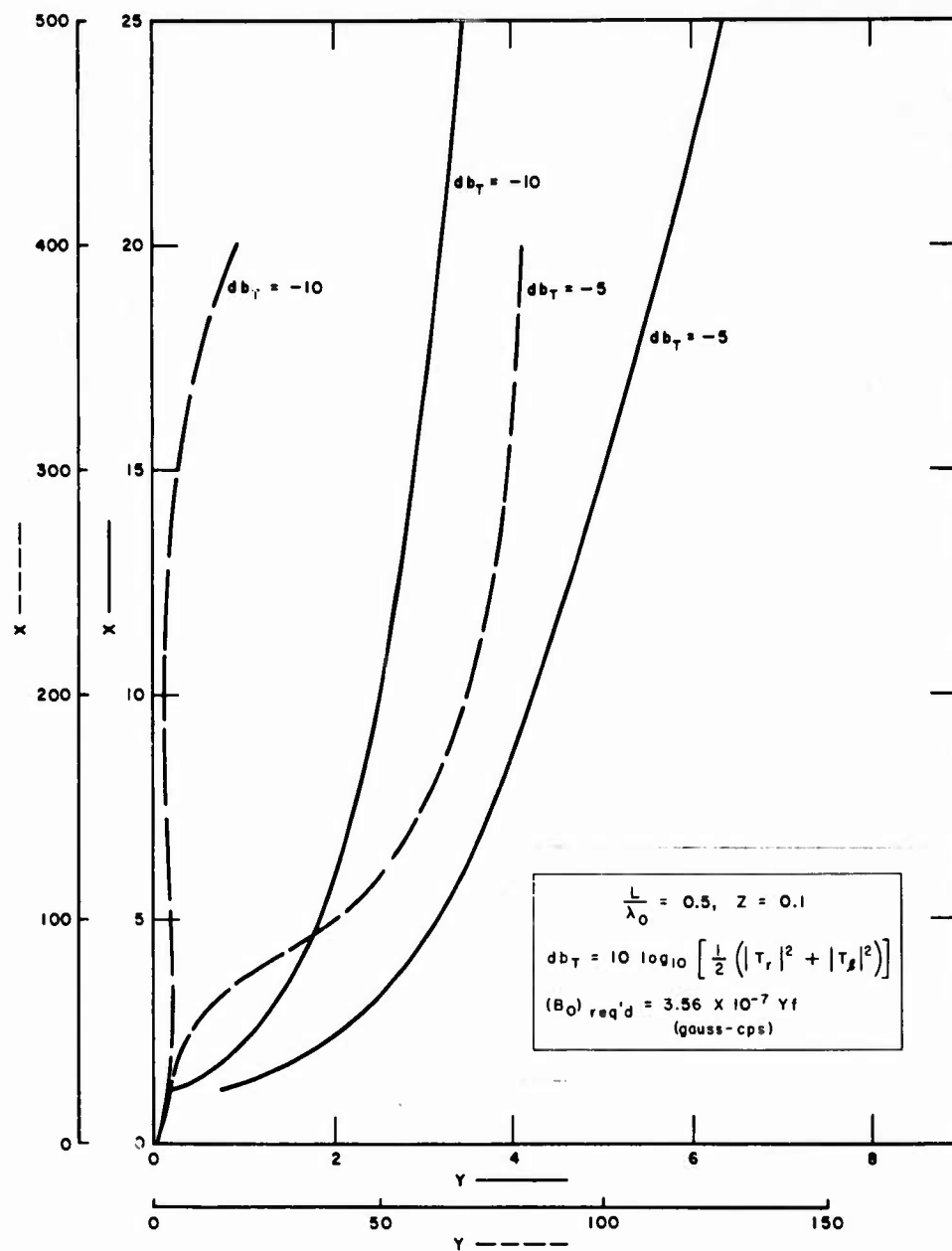
Fig. 6. X versus Y for homogeneous plasma slab; L/λ_0 = 0.5, Z = 0.1
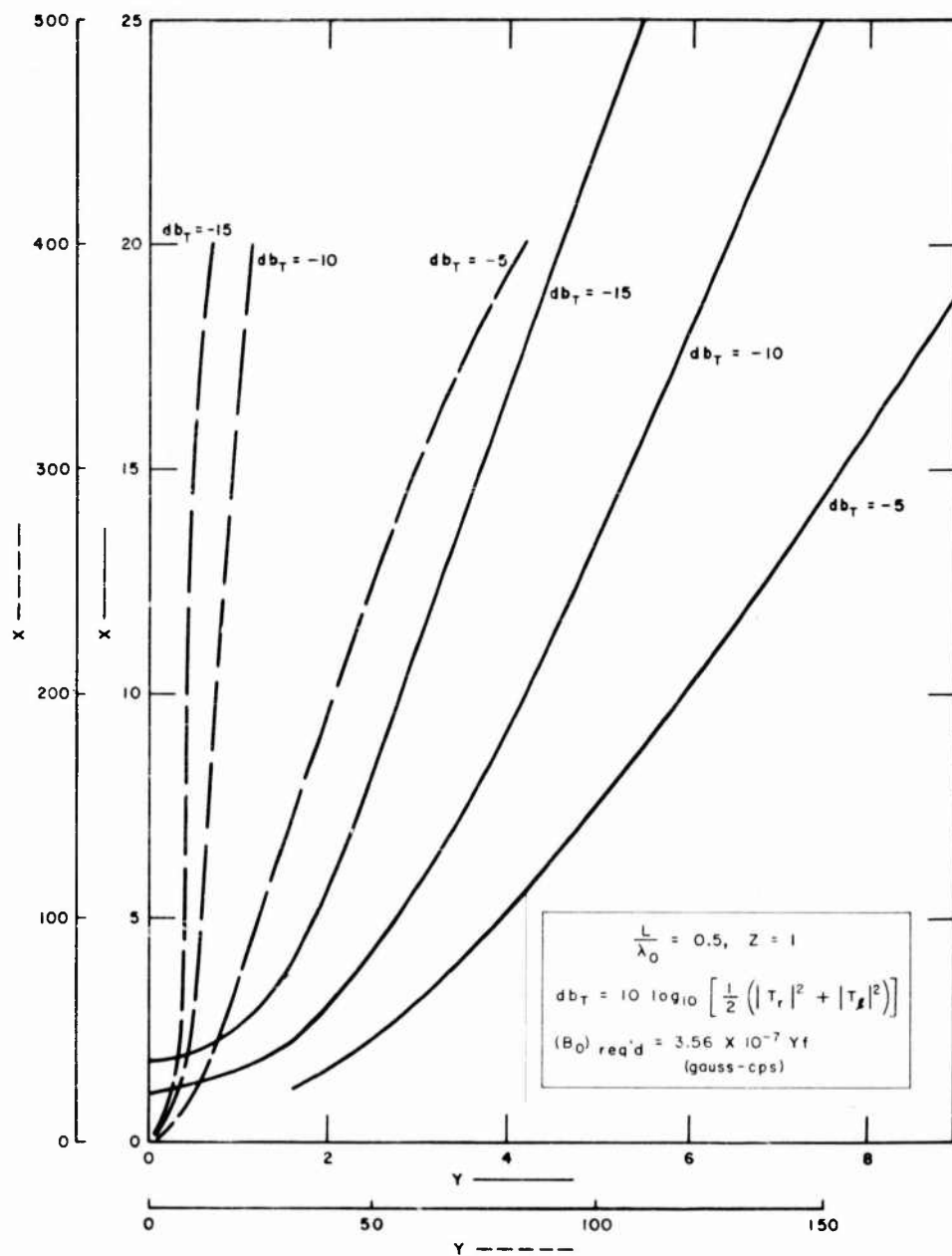
Fig. 7. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 0.5$, $Z = 1$
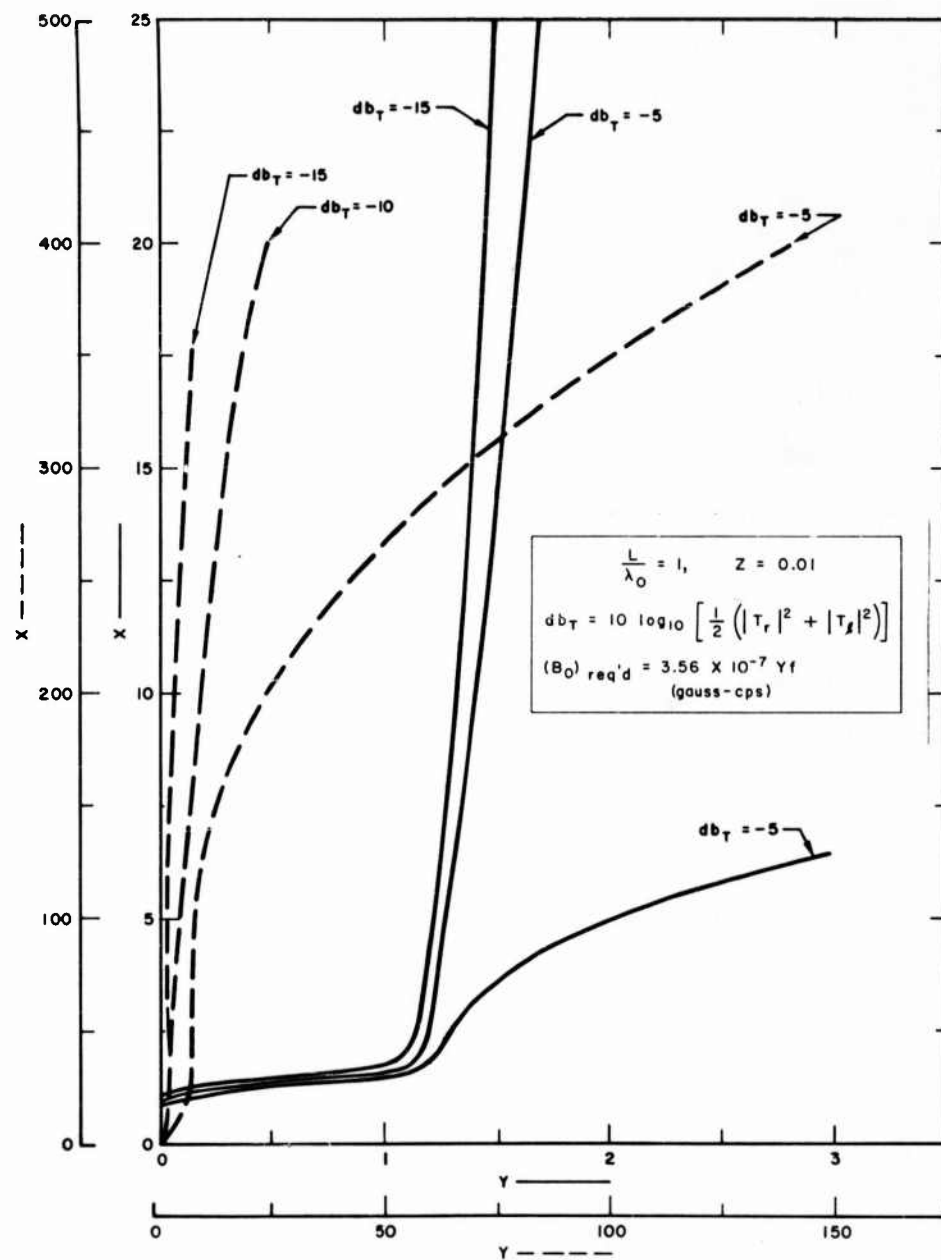
Fig. 8. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 0.01$
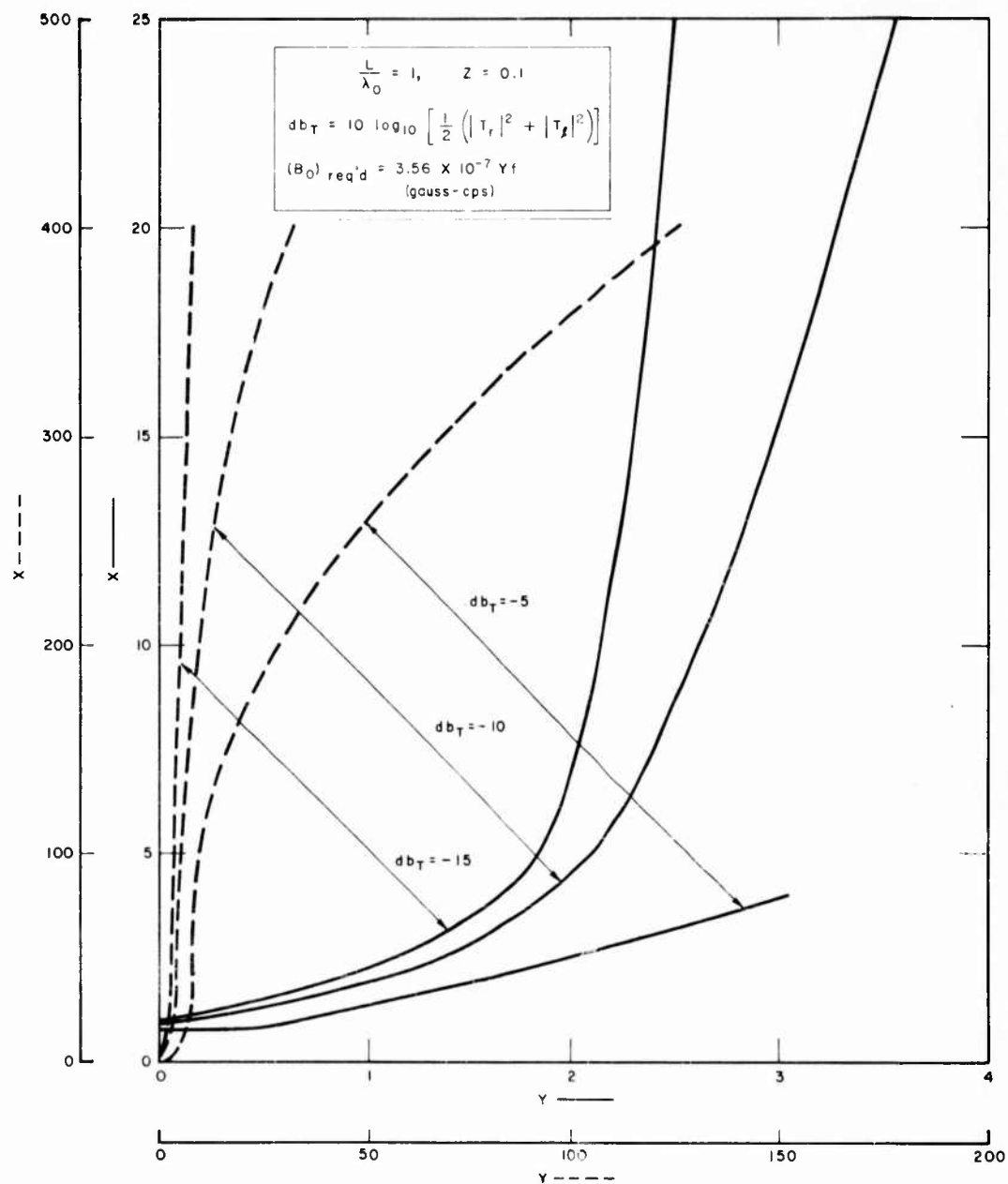
Fig. 9. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 0.1$
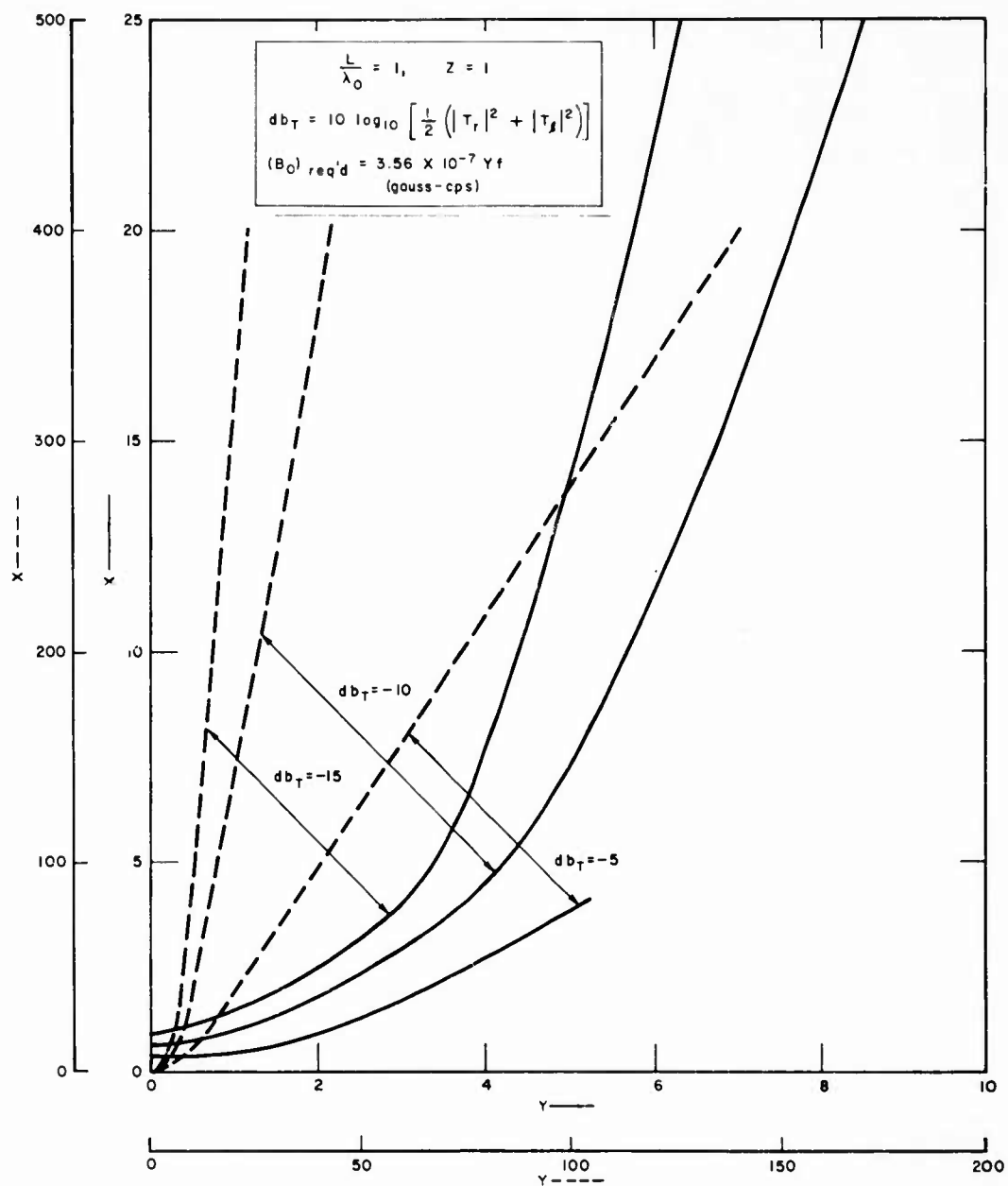
Fig. 10. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 1$, $Z = 1$
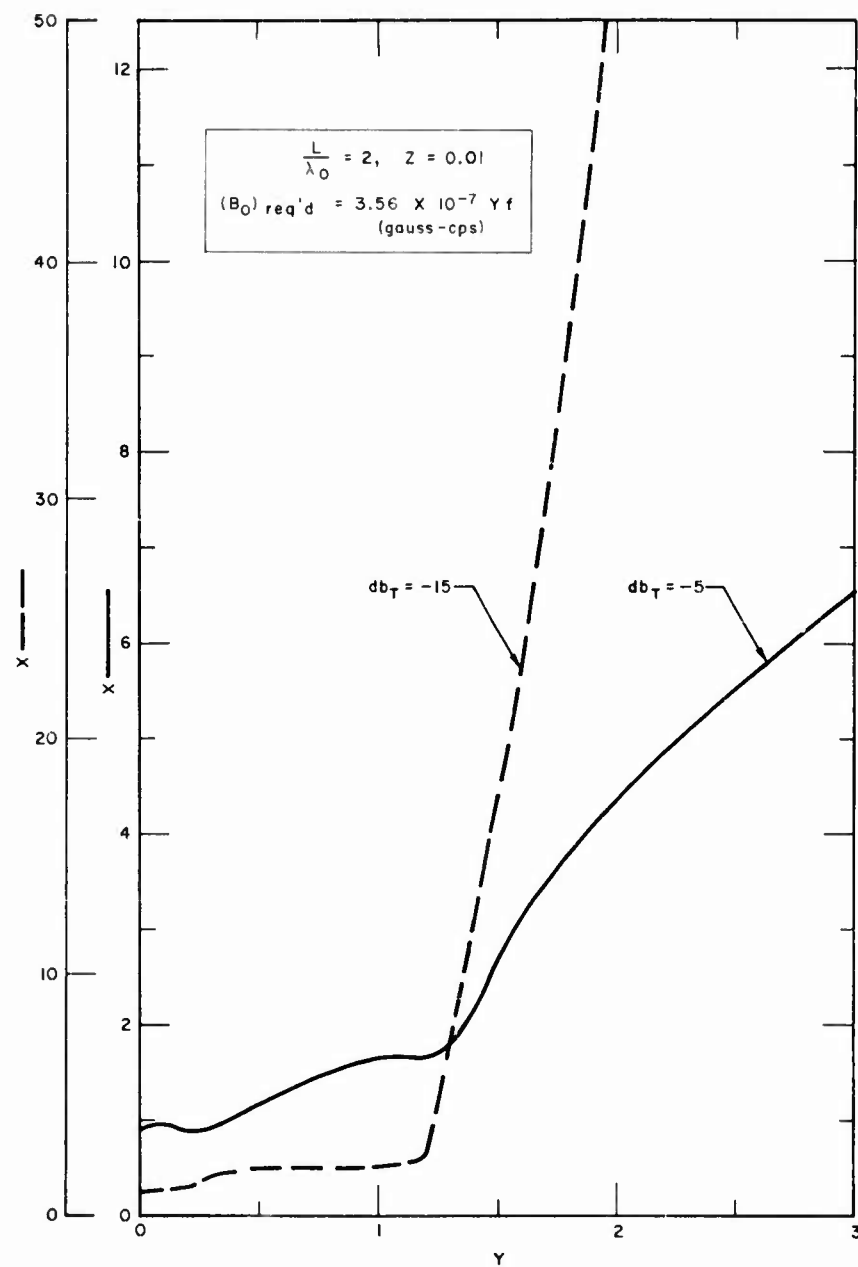
Fig. 11. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 2$, $Z = 0.01$
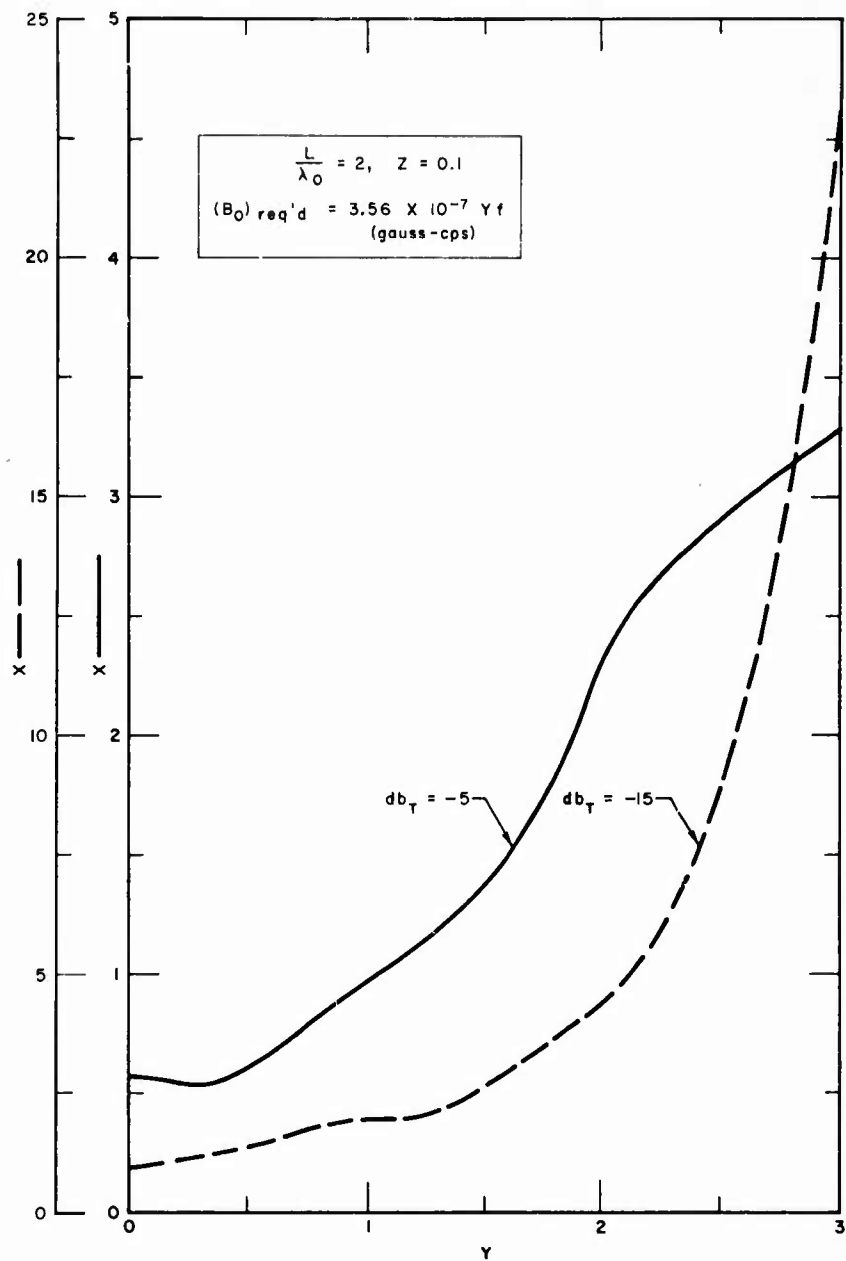
Fig. 12. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 2$, $Z = 0.1$

$\frac{L}{\lambda_0} = 2, \quad Z = 1$

$(B_0)_{req'd} = 3.56 \times 10^{-7} \ Yf$

(gauss-cps)

$db_T = -15$

$db_T = -5$

X

Y

Fig. 13.   X versus Y for homogeneous plasma slab; $L/\lambda_0 = 2$, $Z = 1$

Fig. 14. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 4$, $Z = 0.01$

Fig. 15. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 4$, $Z = 0.1$

Fig. 16. X versus Y for homogeneous plasma slab; $L/\lambda_0 = 4$, $Z = 1$

Case	Alt. (kft)	L (cm)	u (kfps)	f_p	f_c
1	50	0.119	26	5.2×10^{10}	4.2×10^{10}
2			23.6	3.9×10^{10}	3.8×10^{10}
3			22	2.3×10^{10}	3.2×10^{10}
4			20	1.3×10^{10}	2.7×10^{10}
5	100	0.406	26	2.6×10^{10}	4.9×10^{9}
6			23.6	10^{10}	4.2×10^{9}
7			22	6×10^{9}	3.5×10^{9}
8			20	3.2×10^{9}	2.9×10^{9}
9	150	1.25	26	8.6×10^{9}	5.7×10^{8}
10			23.6	2.9×10^{9}	4.7×10^{8}
11			22	1.5×10^{9}	3.9×10^{8}
12			20	8×10^{8}	3×10^{8}

$T_W = 1500^{\circ}K$

Axial position = 11 ft

Fig. 17. Re-entry 10-degree half angle cone at zero angle of attack, definition of cases

B₀ (kgauss)	Case 1	2	3	4	5	6	7	8	9	10	11	12
					$f = 2.4 \times 10^8$ cps							
0	-5.2	-3.55	-1.65	-0.67	-16.8	-6.1	-3.15	-1.2	-25.6	-9.75	-4.0	-1.47
5	-4.8	-3.2	-1.4	-0.5	-7.8	-1.0	-0.28	-0.06	-1.9	-0.05	0	0
10	-4.1	-2.58	-1.0	-0.3	-3.75	-0.3	-0.08	-0.02	-0.52	0	0	0
15	-3.3	-1.95	-0.68	-0.22	-2.1	-0.12	-0.04	0	-0.25	0	0	0
20	-2.6	-1.45	-0.47	-0.12	-1.3	-0.08	-0.02	0	-0.12	0	0	0
100	-0.2	-0.1	-0.03	-0.01	-0.07	0	0	0	0	0	0	0
					$f = 3 \times 10^9$ cps							
0	-5.2	-3.55	-1.6	-0.67	-17.5	-5.05	-2.15	-0.65	-16.5	-1.0	-.22	0
5	-4.8	-3.23	-1.36	-0.5	-7.65	-1.0	-0.3	-0.05	-1.55	-0.04	0	0
10	-4.1	-2.58	-1.0	-0.32	-3.7	-0.3	-0.09	0	-0.45	0	0	0
15	-3.3	-1.96	-0.7	-0.22	-2.1	-0.12	-0.04	0	-0.22	0	0	0
20	-2.58	-1.48	-0.5	-0.13	-1.28	-0.09	-0.02	0	-0.12	0	0	0
100	-0.2	-0.1	-0.03	0	-0.07	0	0	0	0	0	0	0
					$f = 3 \times 10^{10}$ cps							
0	-4.5	-2.72	-1.0	-0.32	-3.43	-0.2	-0.06	0	-0.07	0	0	0
5	-4.37	-2.68	-1.0	-0.35	-3.93	-0.38	-0.2	-0.03	-0.13	0	0	0
10	-3.98	-2.45	-1.0	-0.36	-3.7	-2.35	-1.28	-0.45	-3.0	-1.05	-.1	-.03
15	-3.3	-2.08	-0.83	-0.32	-2.16	-0.45	-0.13	-0.04	-0.12	0	0	0
20	-2.7	-1.6	-0.6	-0.2	-0.8	-0.12	-0.04	-0.01	-0.05	0	0	0
100	-0.22	-0.11	0	0	0	0	0	0	-0.01	0	0	0

Fig. 18. Re-entry 10-degree cone at zero angle of attack, homogeneous plasma slab calculations, $db_T = 10 \log_{10}\left[(1/2)\left(|T_r|^2 + |T_\ell|^2\right)\right]$
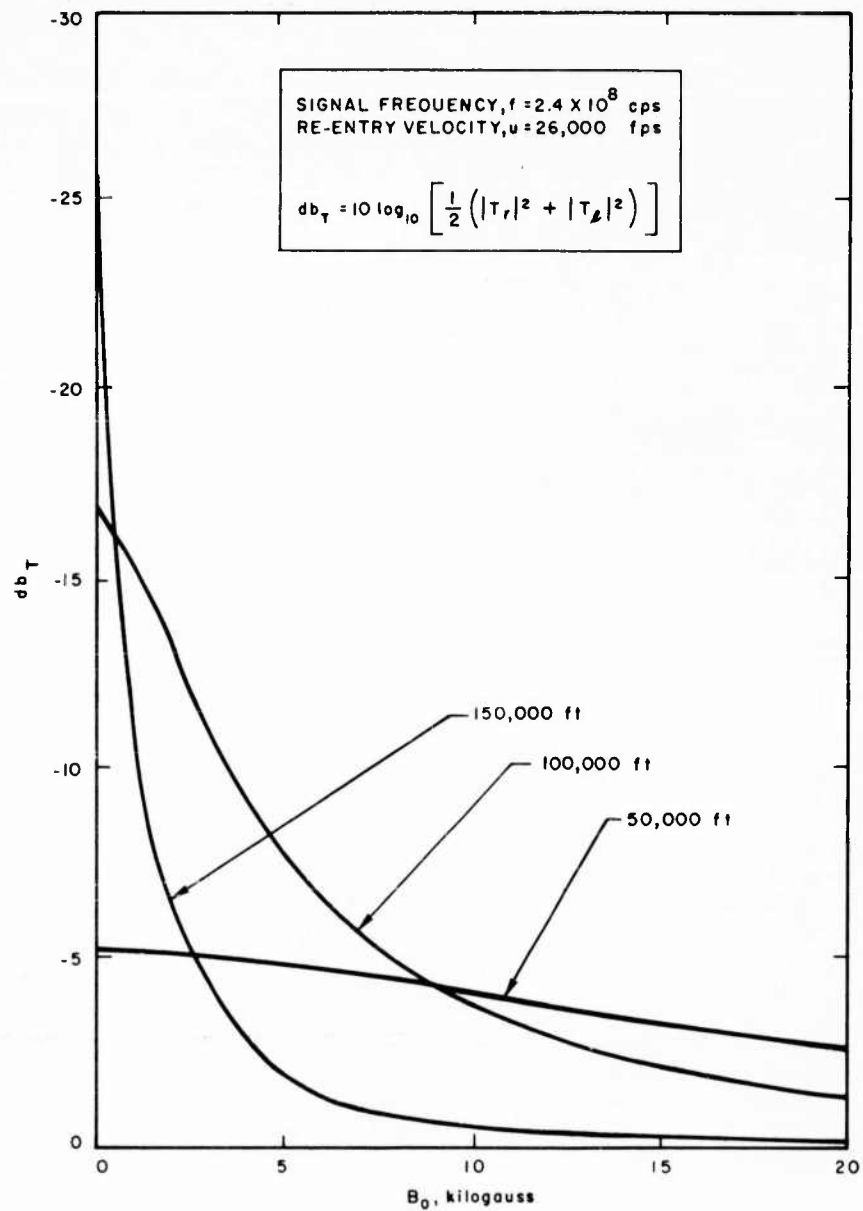
Fig. 19. Transmitted energy, re-entry 10-degree cone at zero angle of attack; $f = 2.4 \times 10^8$ cps, $u = 26,000$ fps

-91-
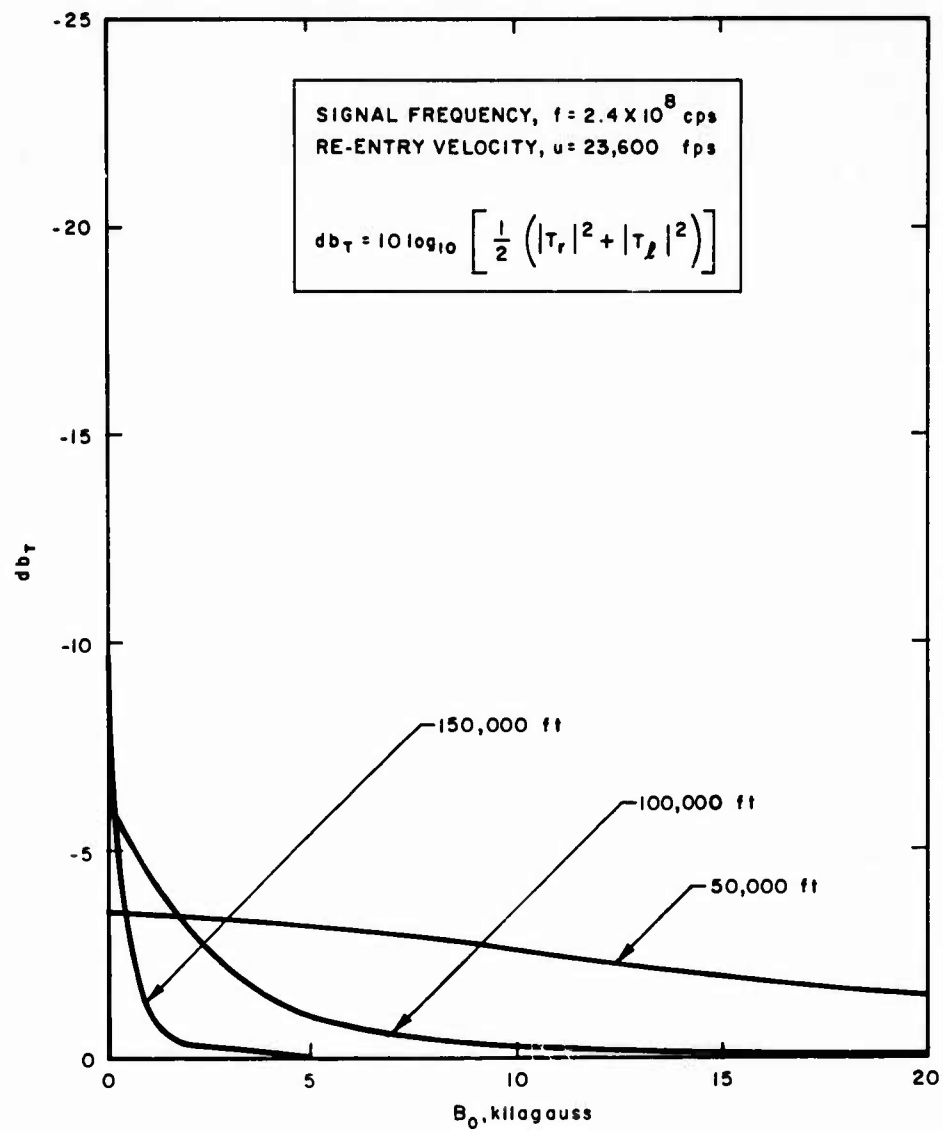
Fig. 20. Transmitted energy, re-entry 10-degree cone at zero angle of attack; $f = 2.4 \times 10^8$ cps, $u = 23,600$ fps
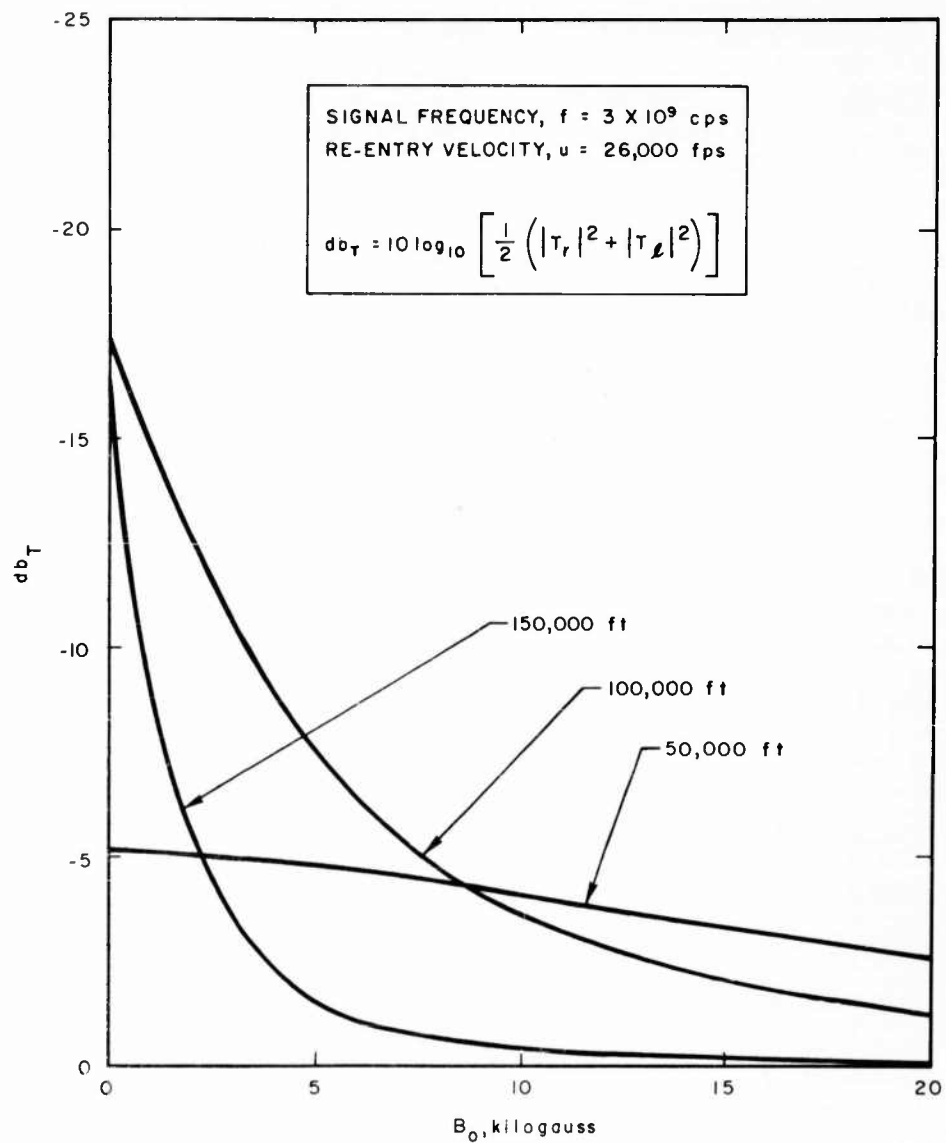
Fig. 21. Transmitted energy, re-entry 10-degree cone at zero angle of attack;
$f = 3 \times 10^9$ cps, u = 26,000 fps
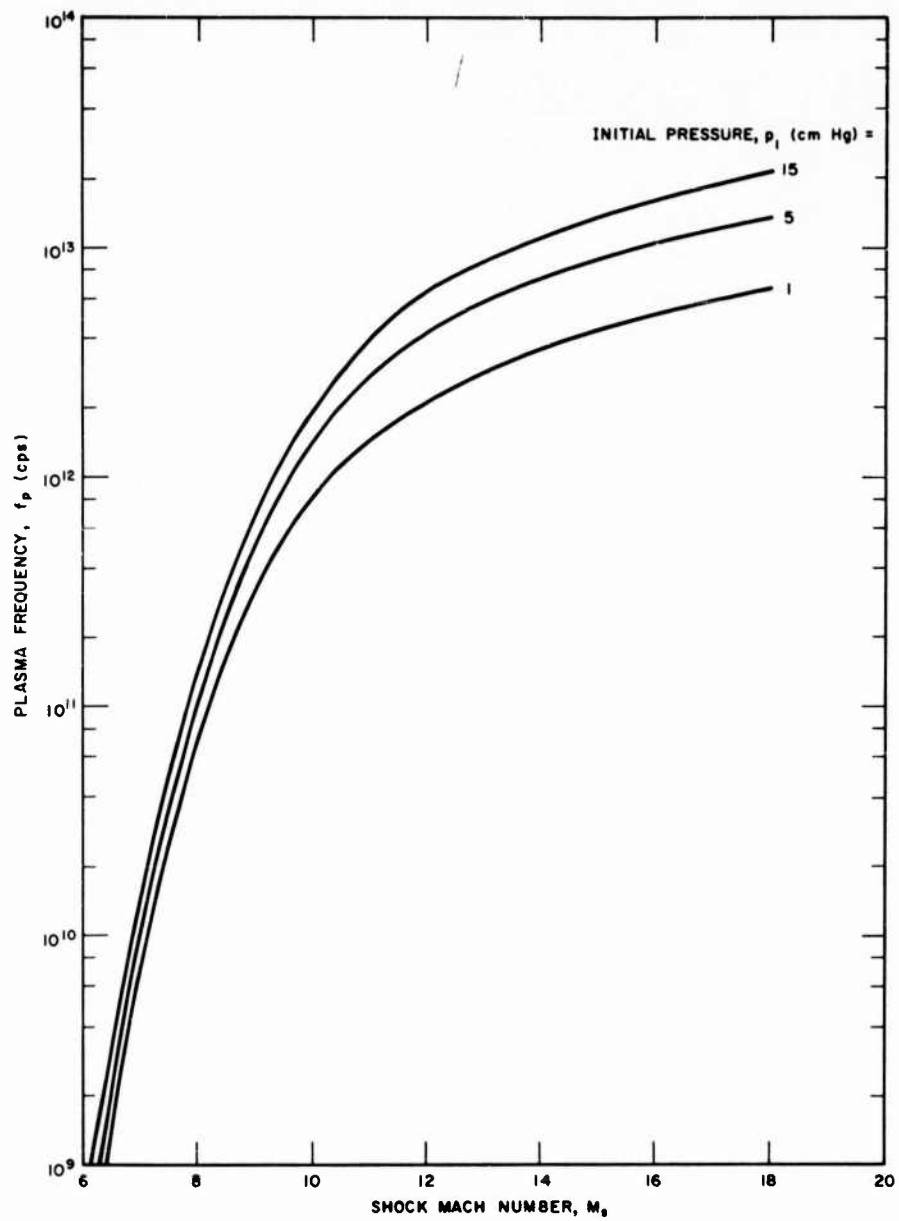
Fig. 22. Shock-tube generated plasma properties (argon);
plasma frequency, f_p (cps)
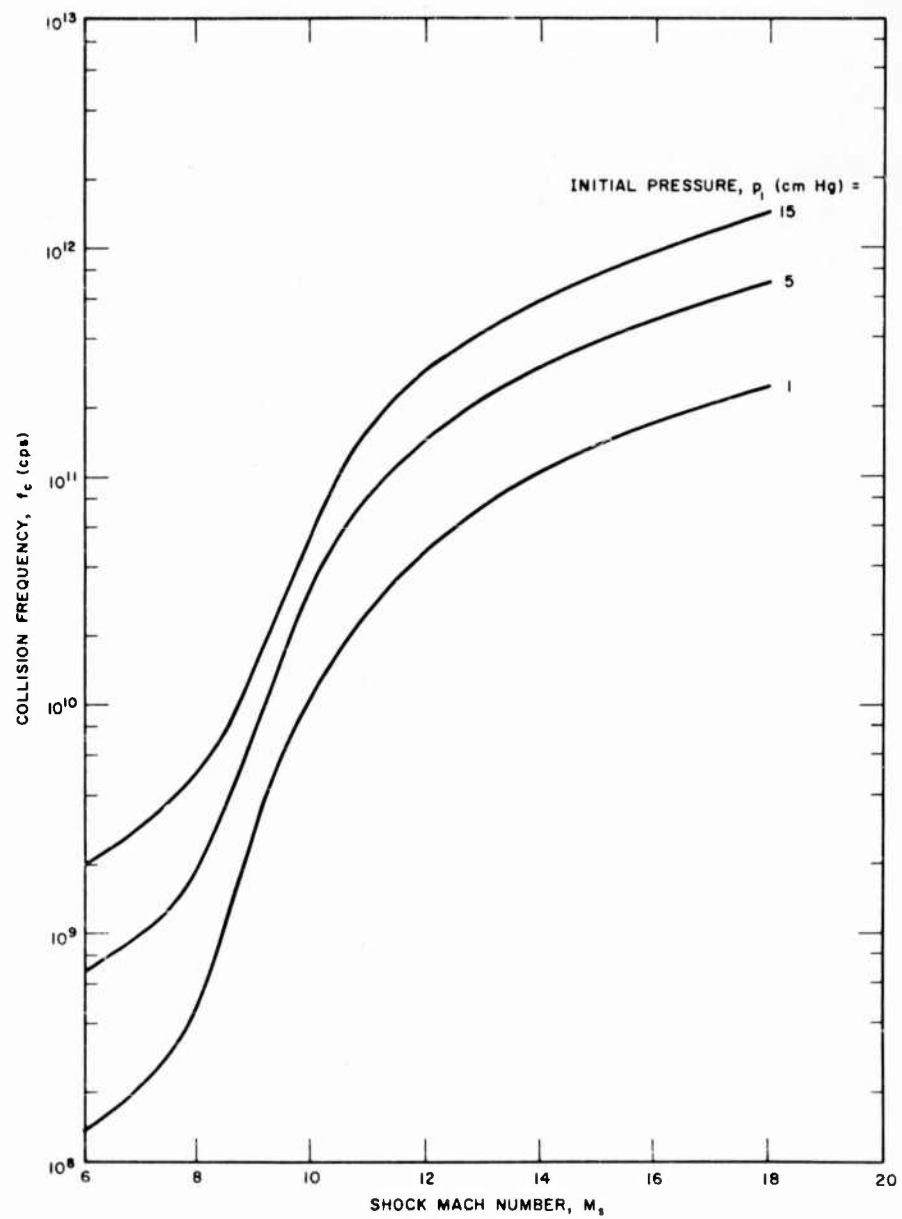
Fig. 23. Shock-tube generated plasma properties (argon);
collision frequency, f_c (cps)
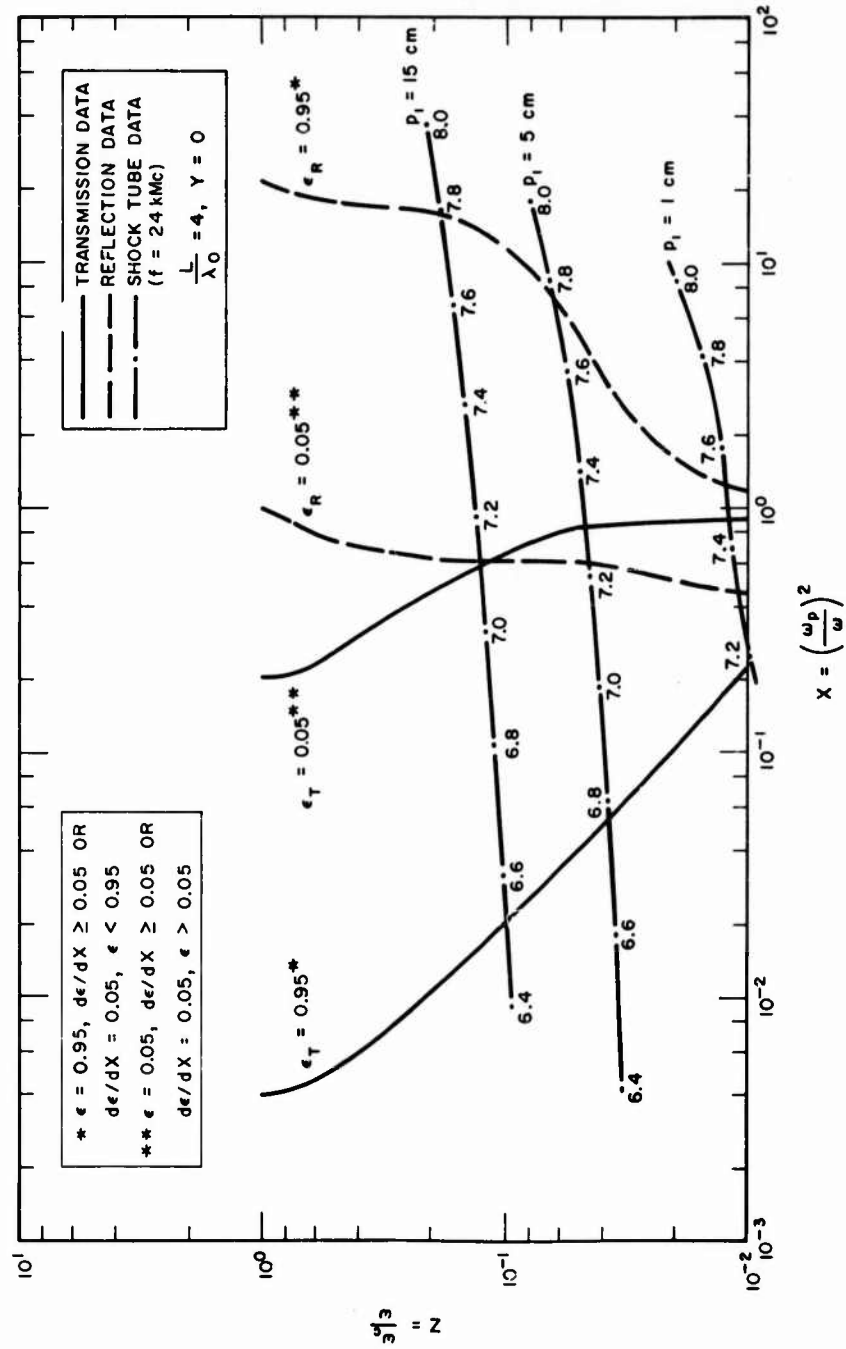
Fig. 24. Homogeneous plasma slab calculations, Z versus X, for shock tube experiment;
$L/\lambda_0 = 4$, $Y = 0$
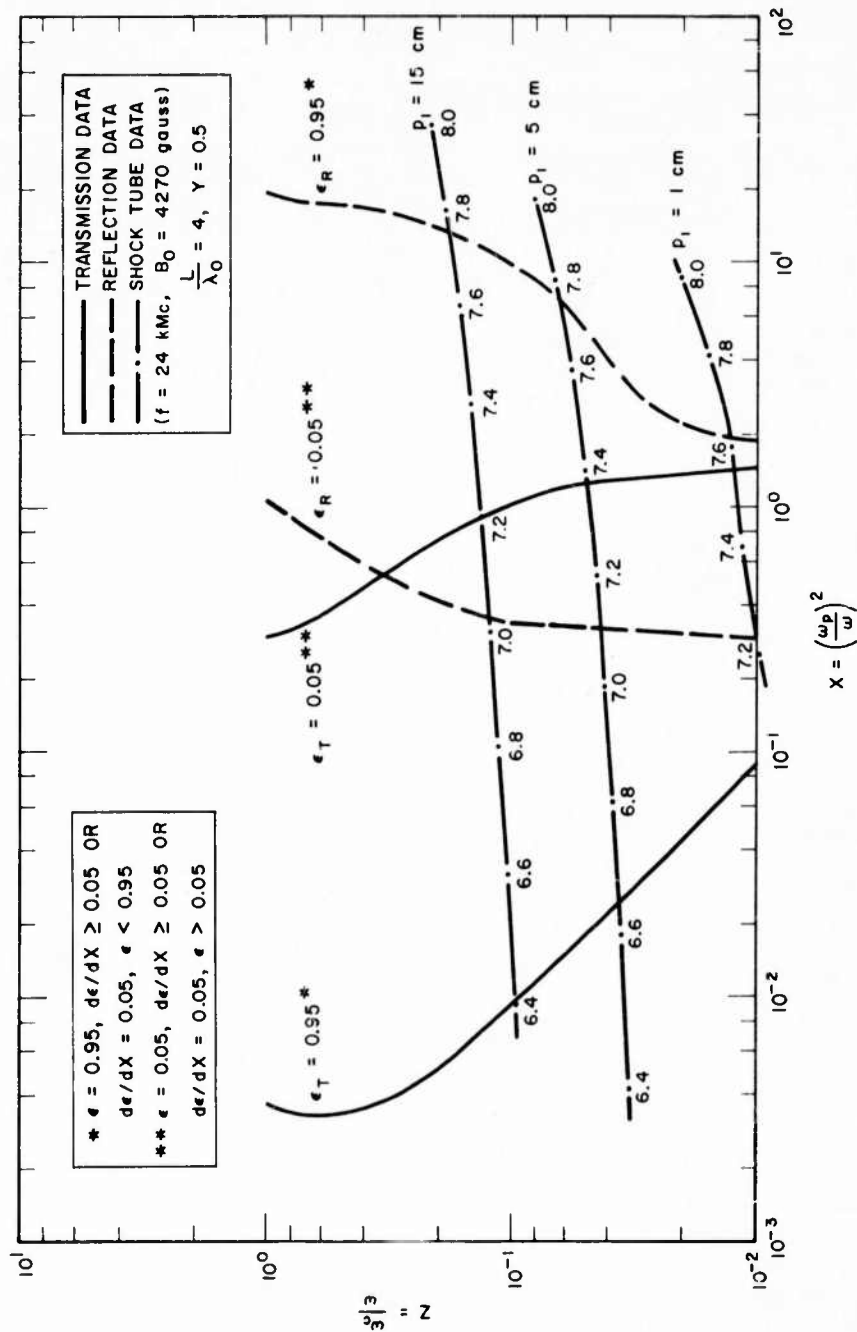
Fig. 25. Homogeneous plasma slab calculations, Z versus X, for shock tube experiment;
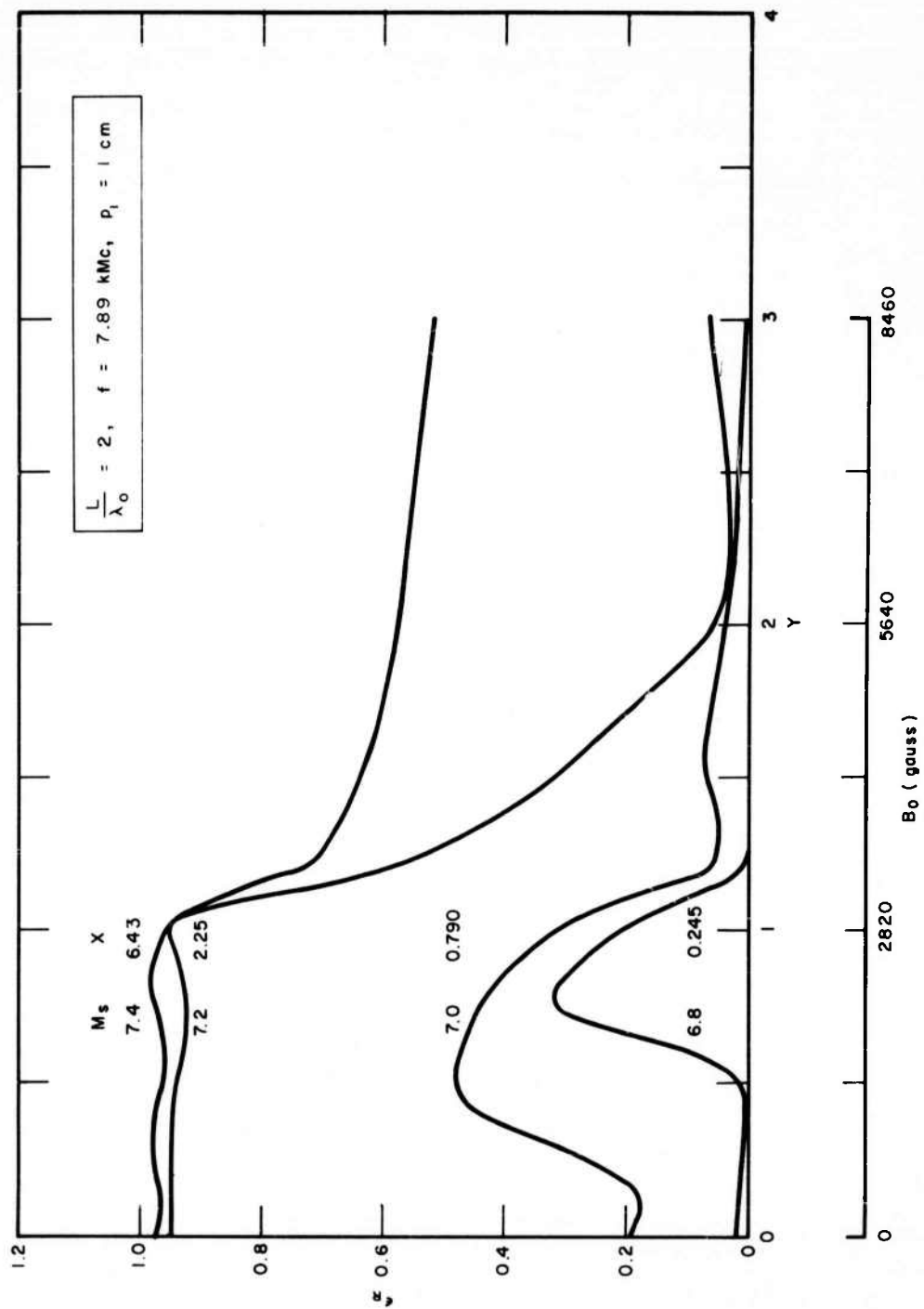
$L/\lambda_0 = 4$, $Y = 0.5$
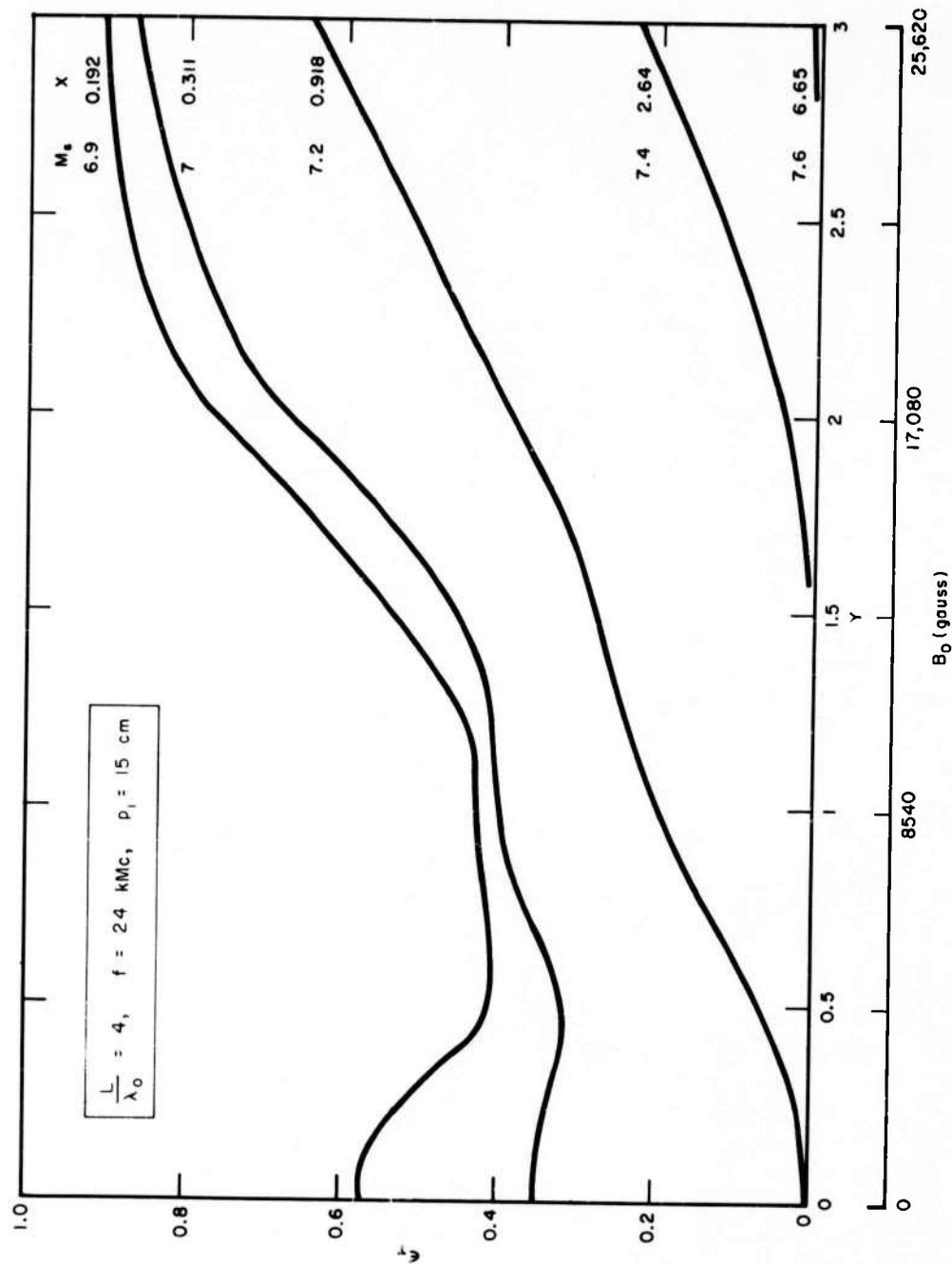
Fig. 26. Homogeneous plasma slab calculations, ϵ_T versus Y, for shock tube experiment;

$L/\lambda_0 = 4$, $f = 24$ kMc, $p_1 = 15$ cm

-98-
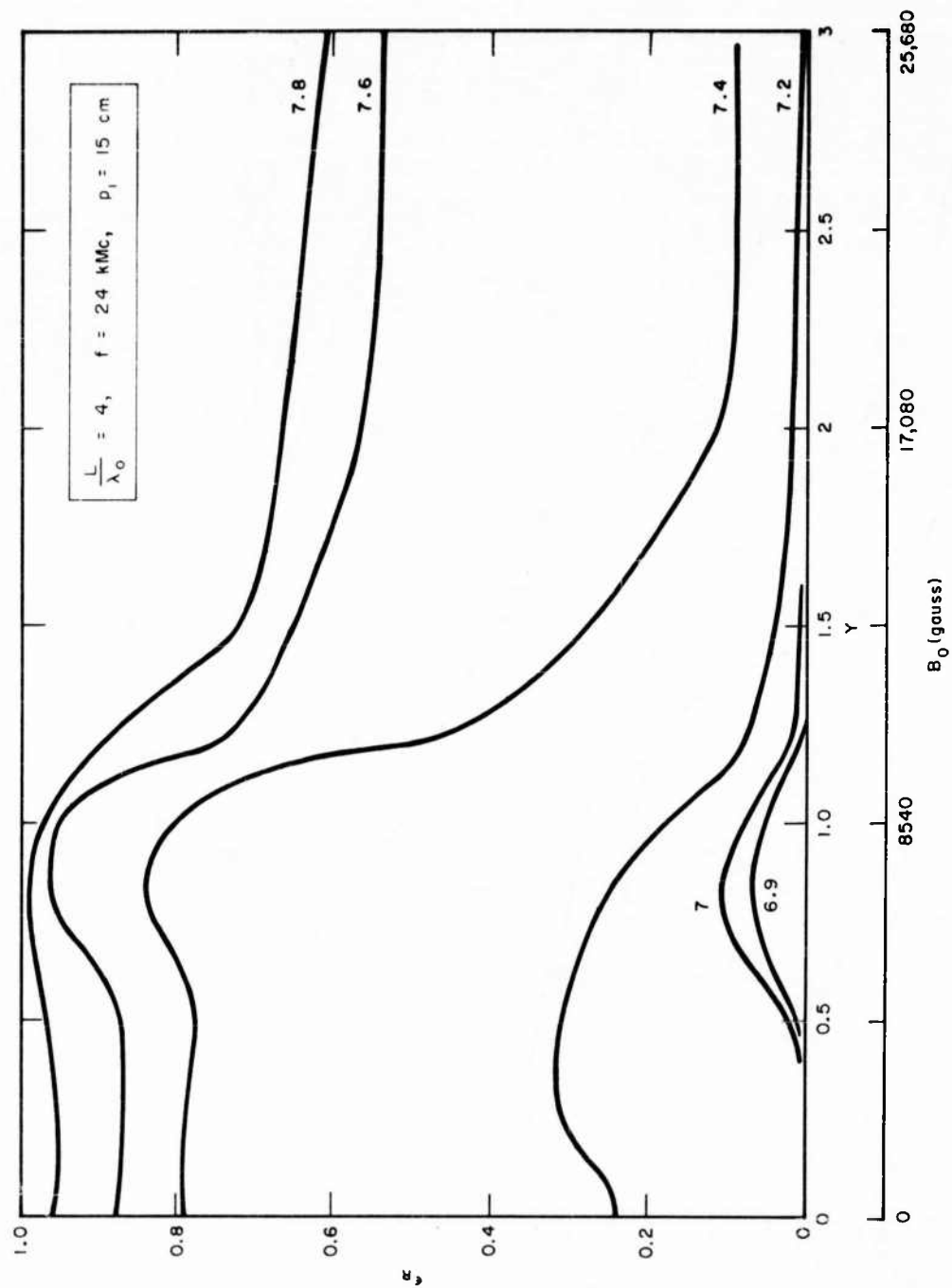
Fig. 27. Homogeneous plasma slab calculations, ϵ_R versus Y, for shock tube experiment;
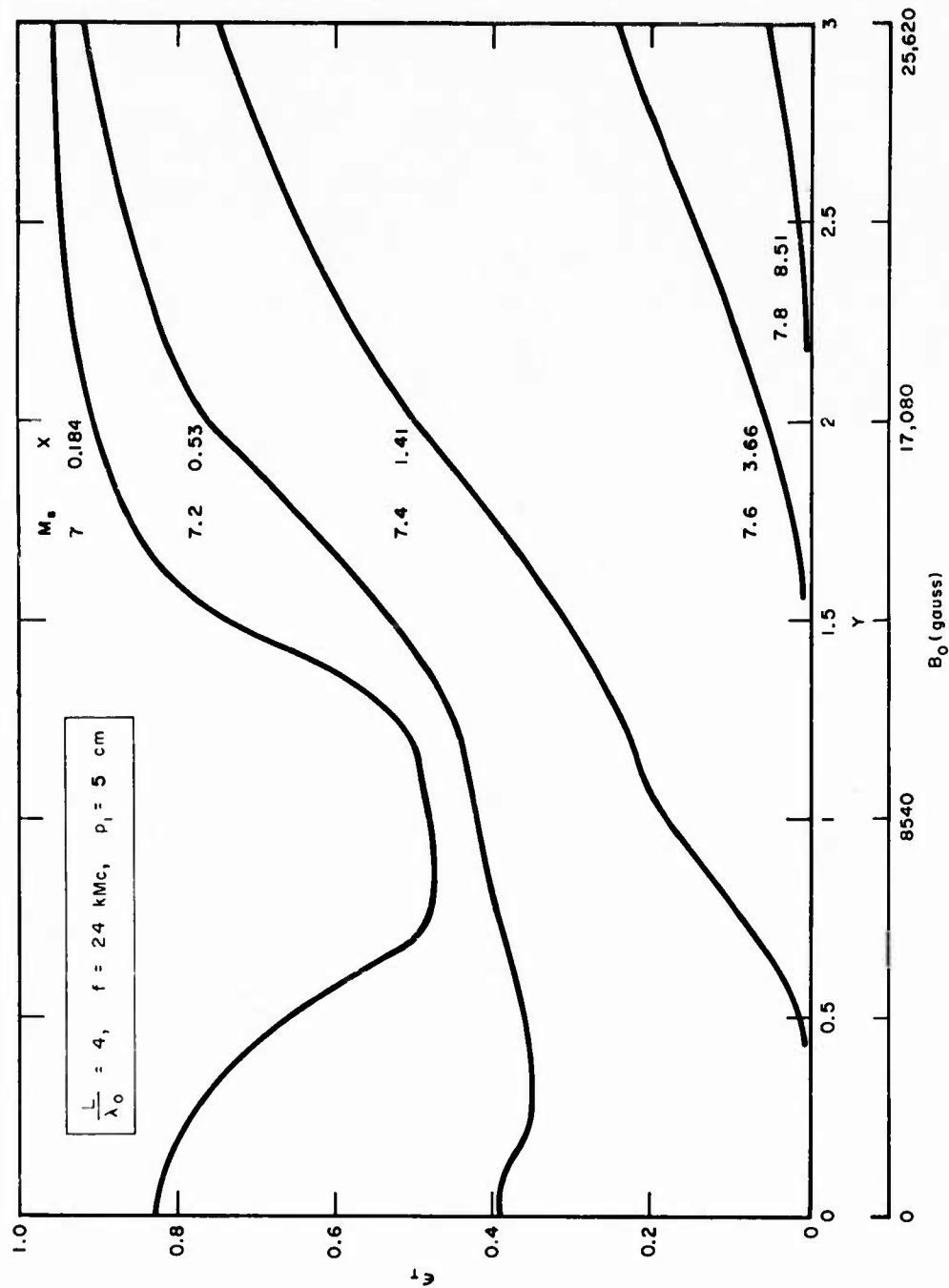
$L/\lambda_0 = 4$, $f = 24$ kMc, $p_1 = 15$ cm

Fig. 28. Homogeneous plasma slab calculations, ϵ_T versus Y, for shock tube experiment;
$L/\lambda_0 = 4$, $f = 24$ kMc, $p_1 = 5$ cm
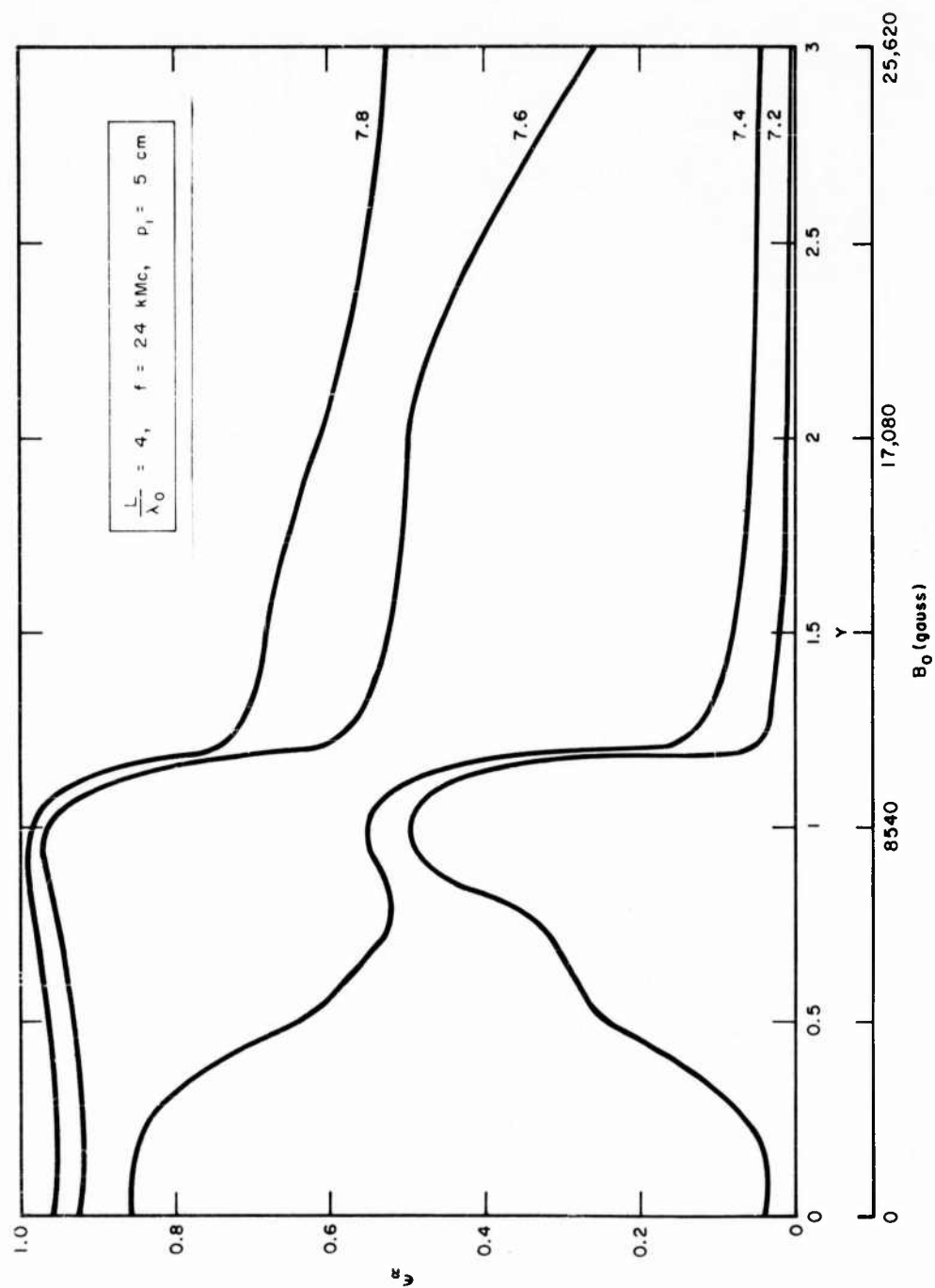
Fig. 29. Homogeneous plasma slab calculations, ϵ_R versus Y, for shock tube experiment;

$L/\lambda_0 = 4$, $f = 24$ kMc, $p_1 = 5$ cm
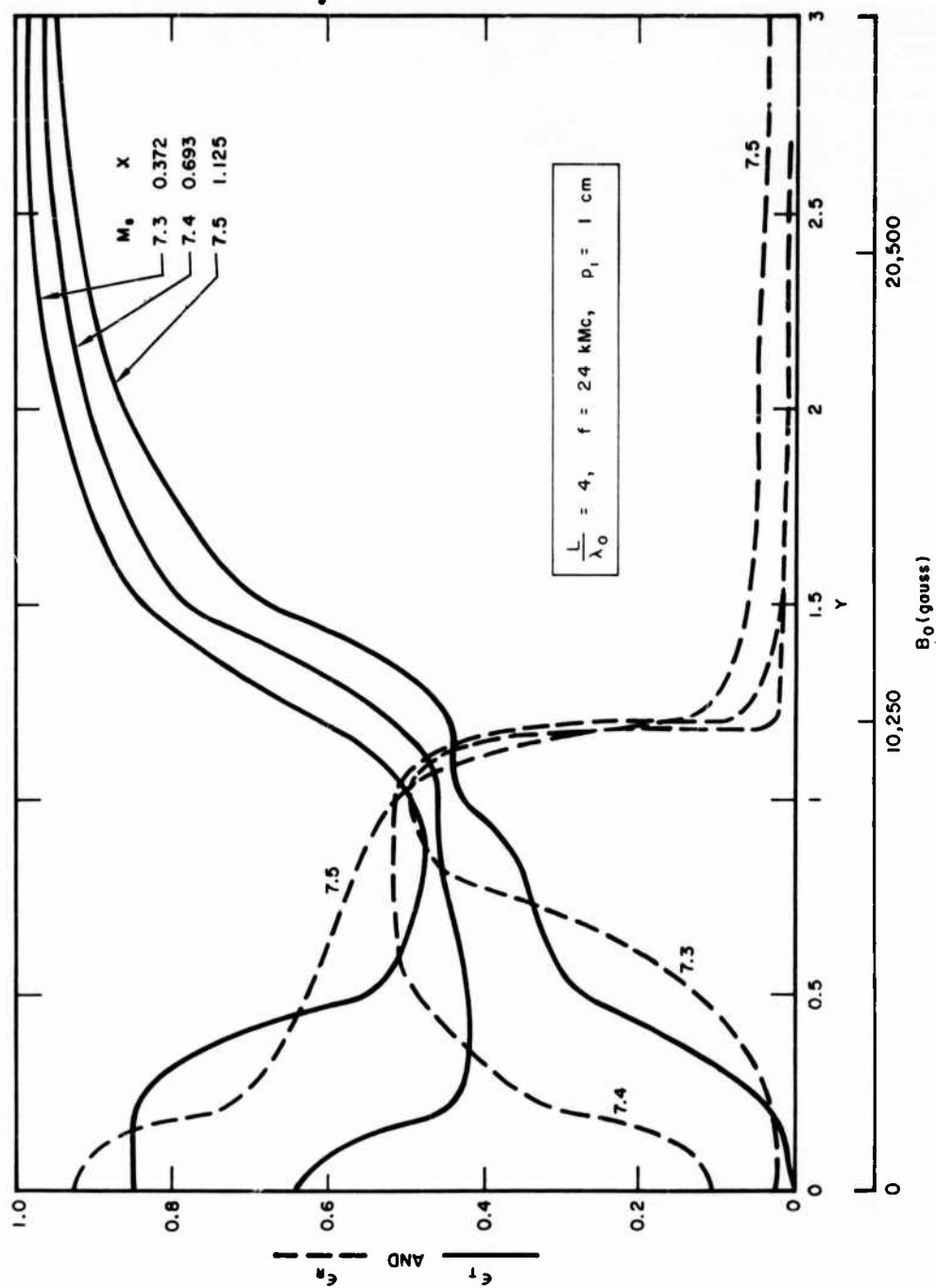
Fig. 30. Homogeneous plasma slab calculations, ϵ_T and ϵ_R versus Y, for shock tube experiment
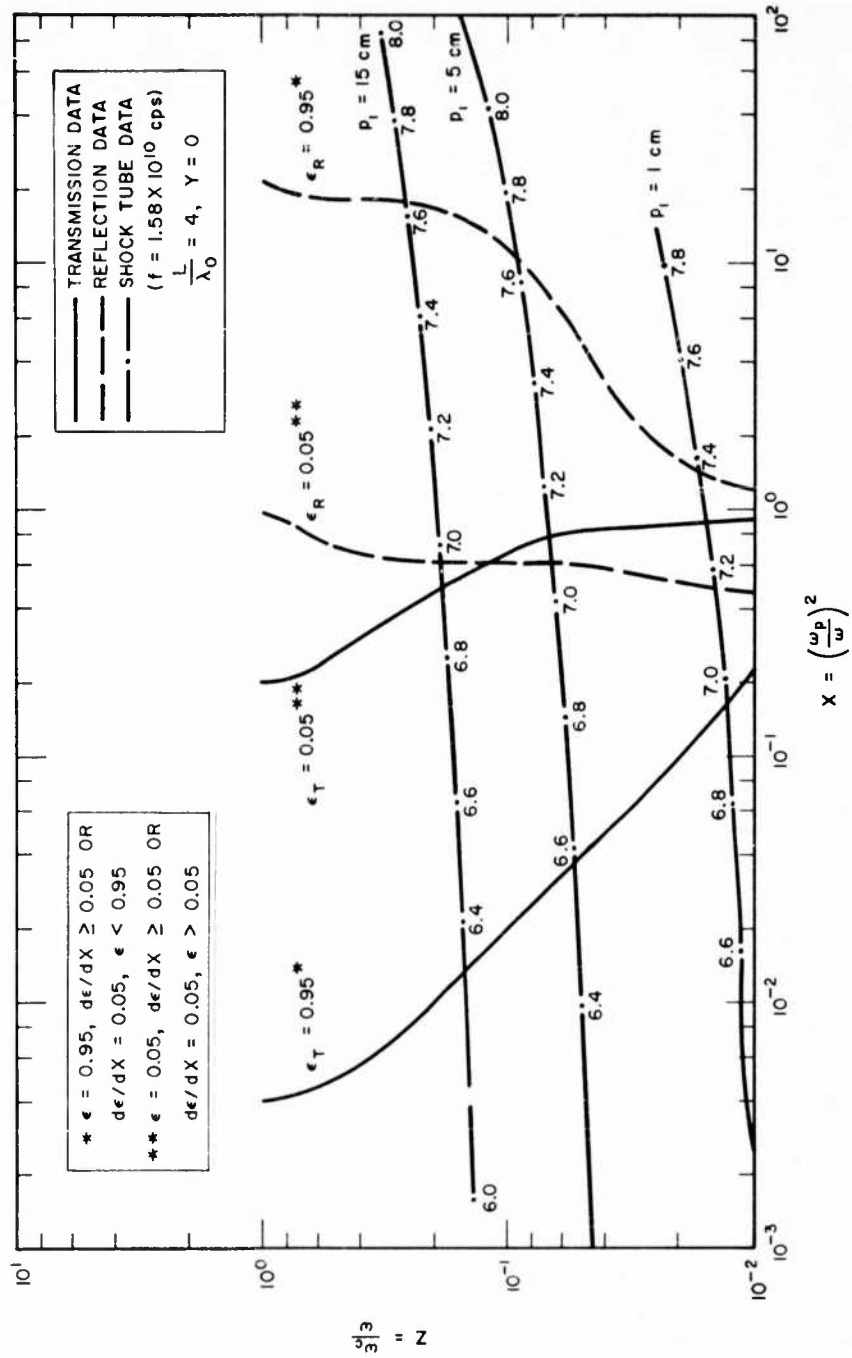
Fig. 31. Homogeneous plasma slab calculations, Z versus X, for shock tube experiment;
$L/\lambda_0 = 4$, $Y = 0$
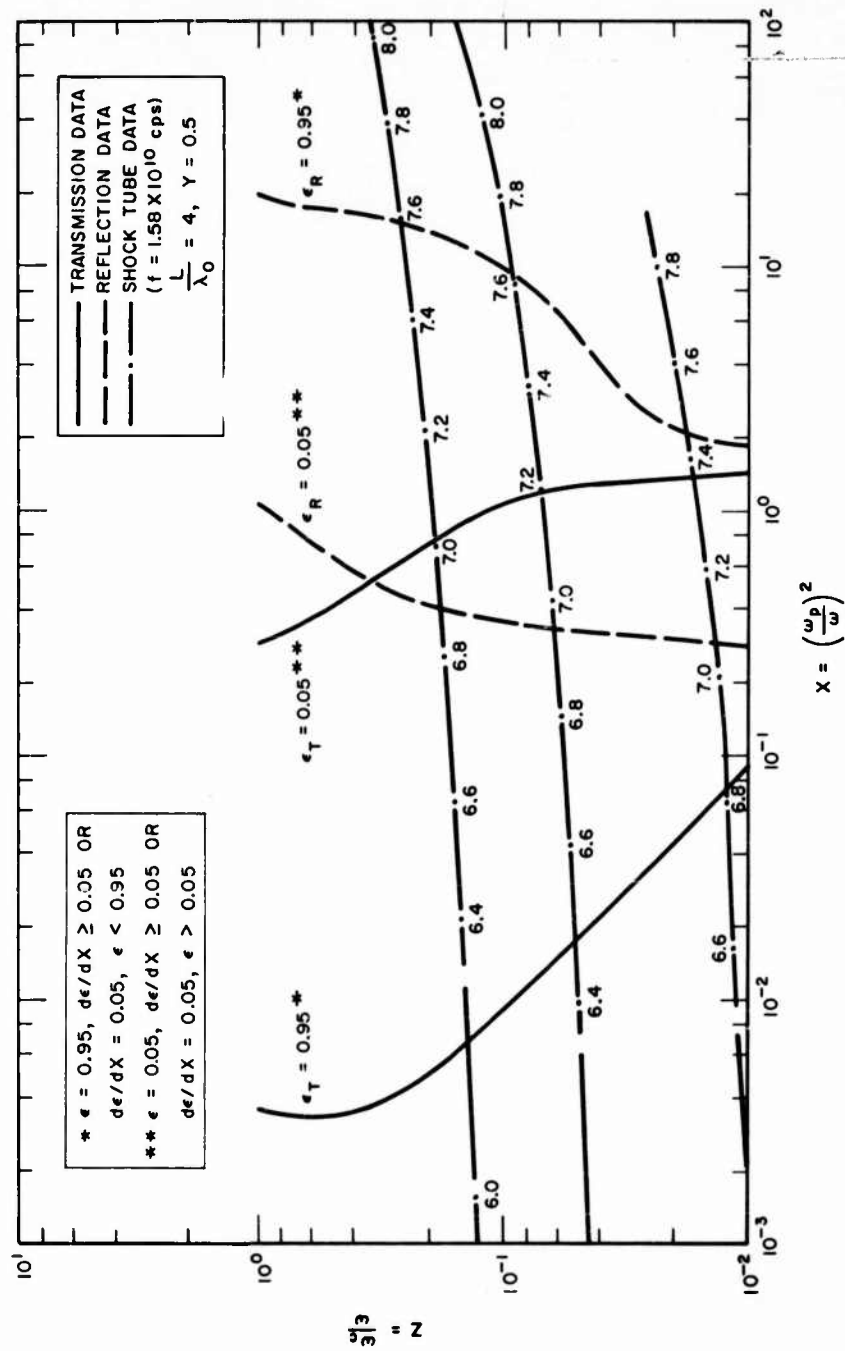
Fig. 32. Homogeneous plasma slab calculations, Z versus X, for shock tube experiment.

$L/\lambda_0 = 4$, $Y = 0.5$
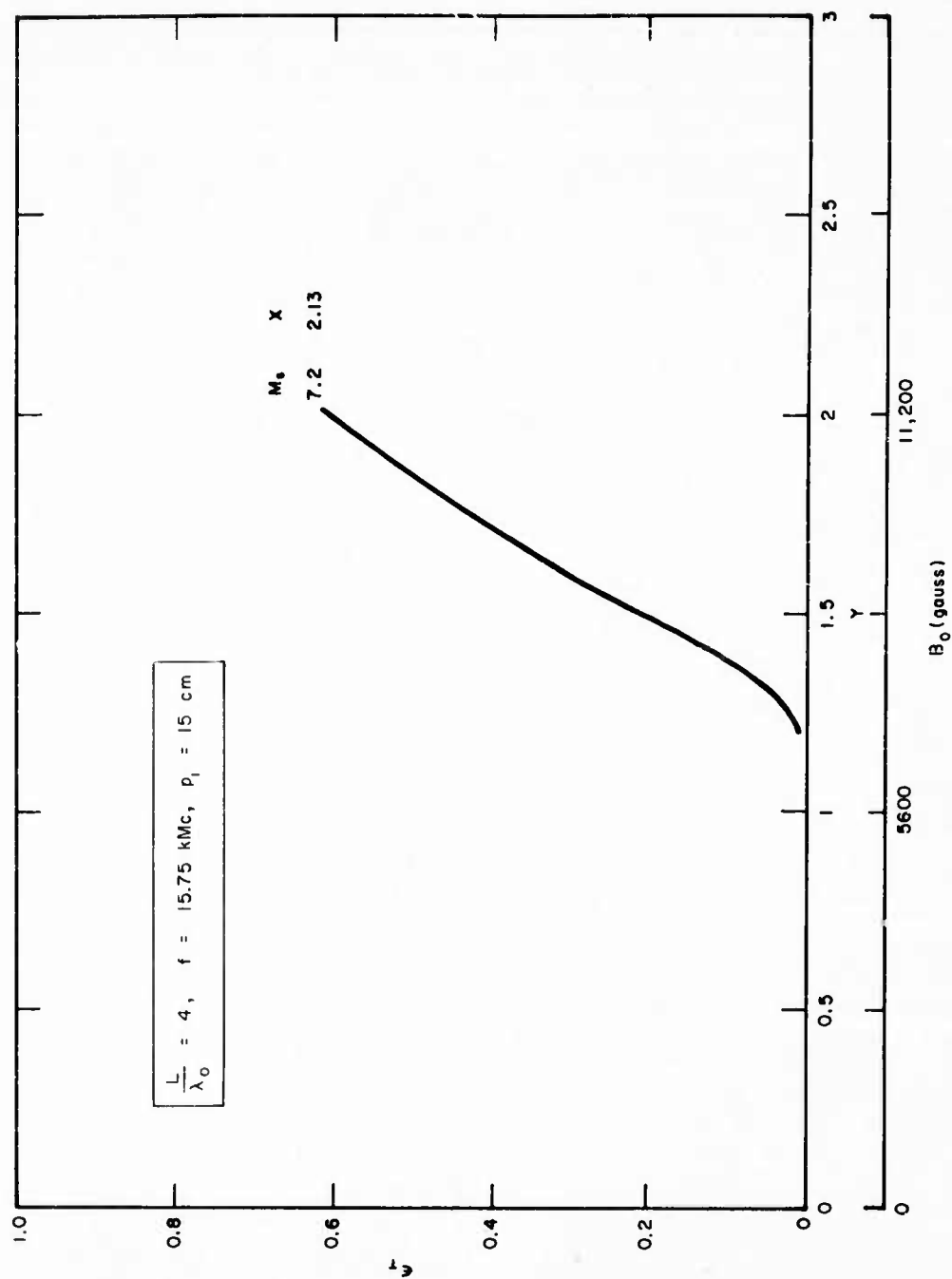
Fig. 33. Homogeneous plasma slab calculations, ϵ_T versus Y, for shock tube experiment;

$L/\lambda_0 = 4$, $f = 15.75$ kMc, $p_1 = 15$ cm
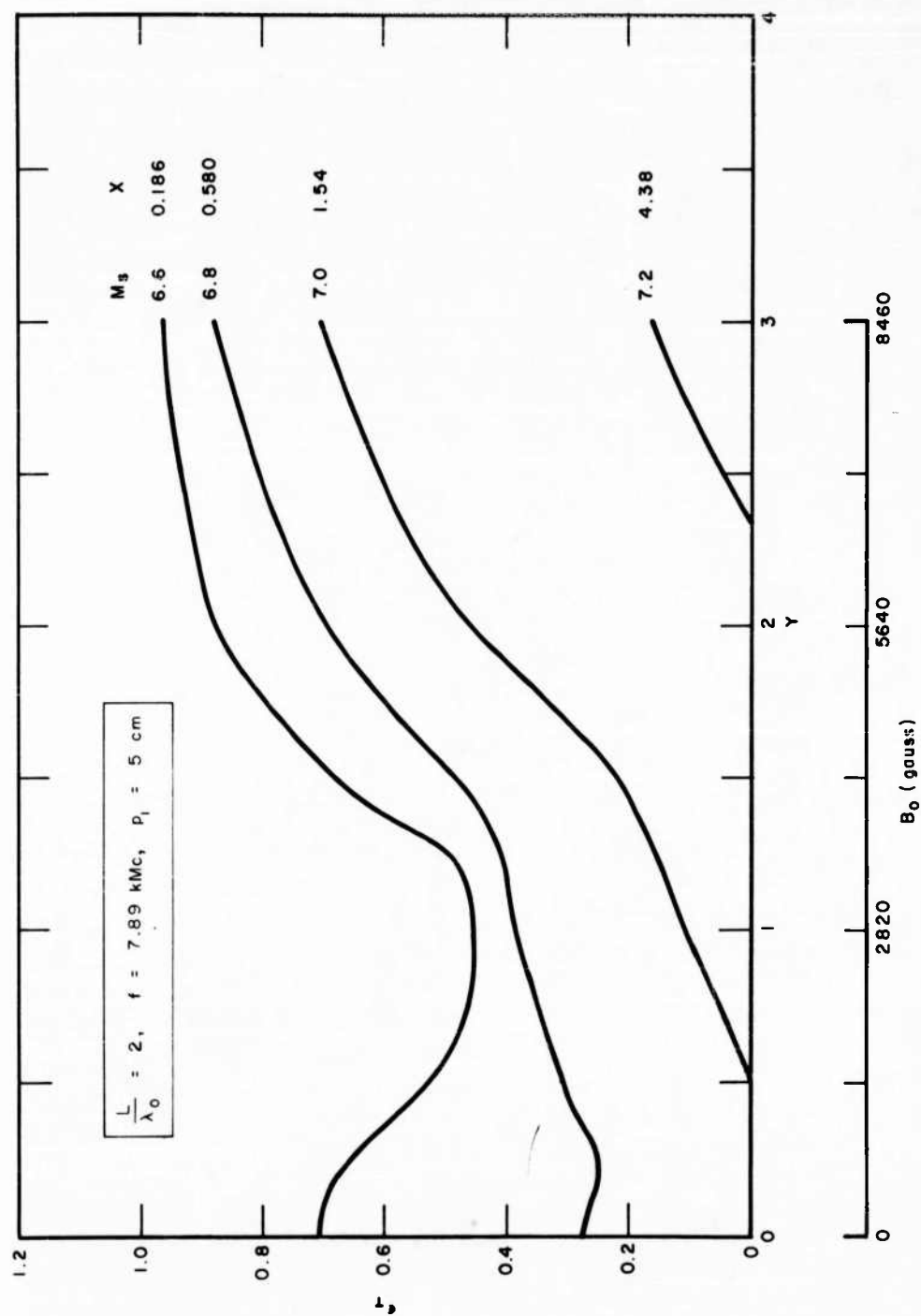
Fig. 34. Homogeneous plasma slab calculations, ϵ_T versus Y, for shock tube experiment:
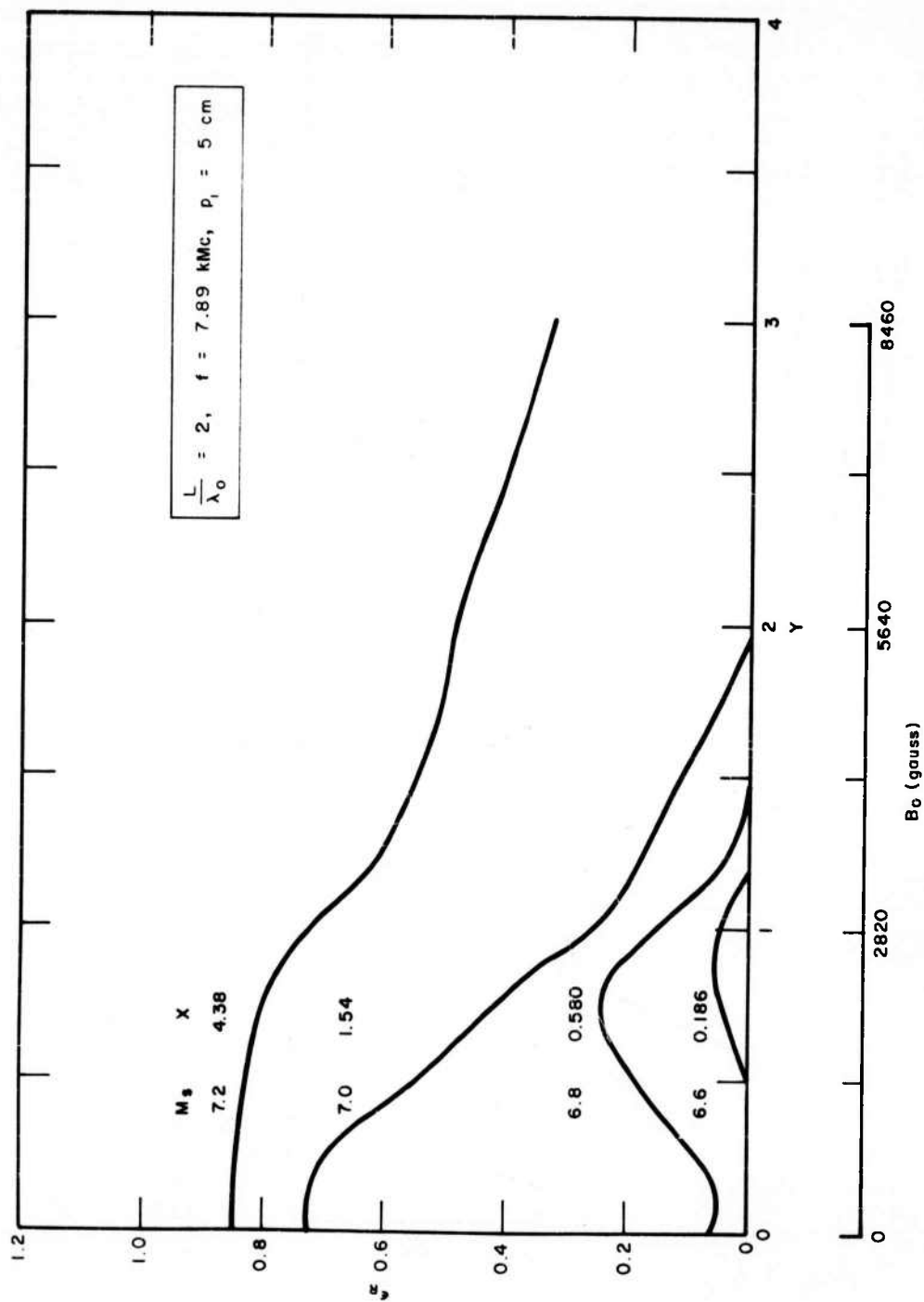
$\frac{L}{\lambda_0} = 2$, $f = 7.89$ kMc, $p_1 = 5$ cm

Fig. 35. Homogeneous plasma slab calculations, ϵ_R versus Y, for shock tube experiment;
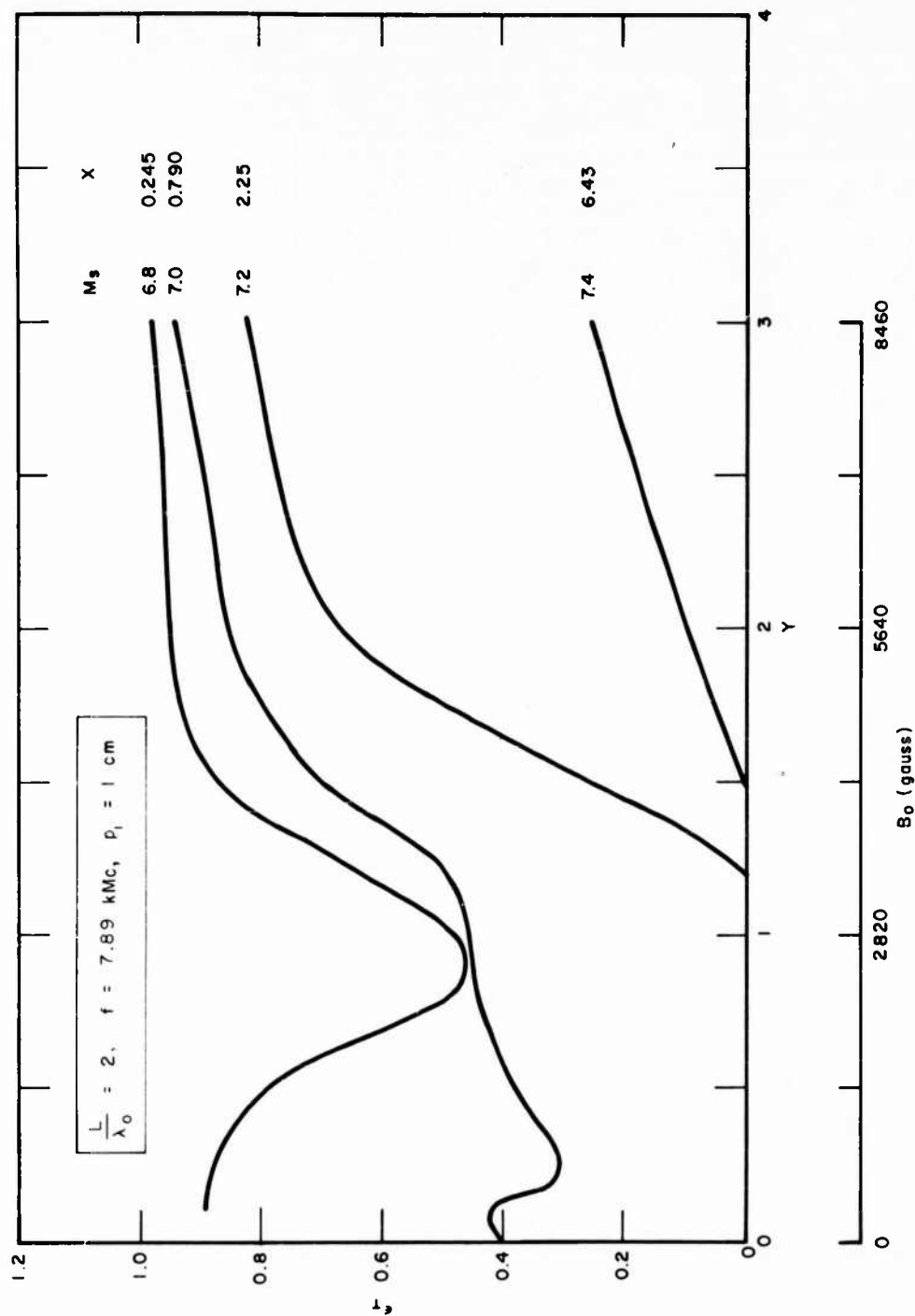
$L/\lambda_0 = 2$, $f = 7.89$ kMc, $p_1 = 5$ cm

Fig. 36. Homogeneous plasma slab calculations, ϵ_T versus Y, for shock tube experiment;

$L/\lambda_0 = 2$, $f = 7.89$ kMc, $p_1 = 1$ cm

Fig. 37. Homogeneous plasma slab calculations, ϵ_R versus Y, for shock tube experiment;
$L/\lambda_0 = 2$, $f = 7.89$ kMc, $p_1 = 1$ cm
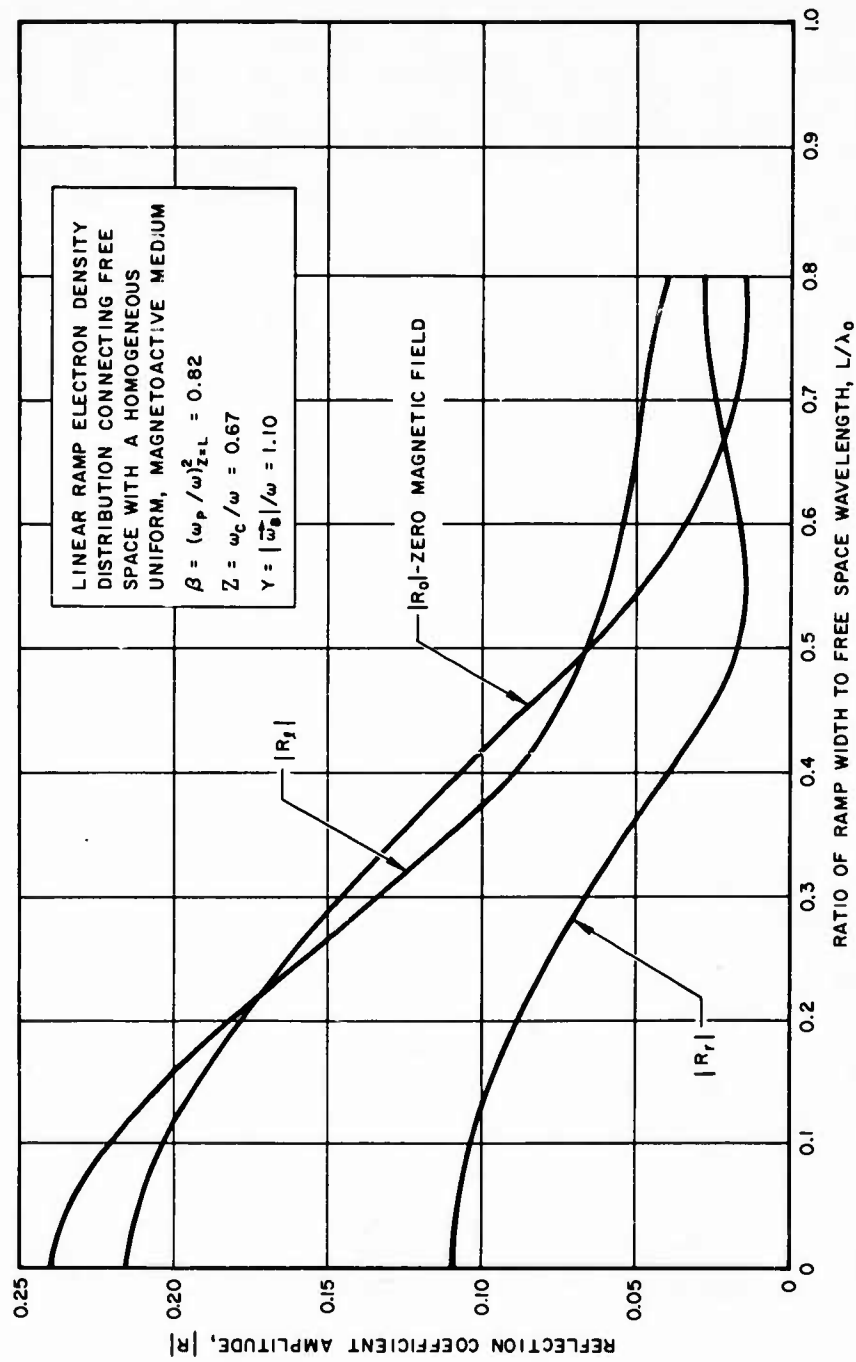
Fig. 38. Reflection coefficient amplitude versus ratio of ramp width to free space wavelength
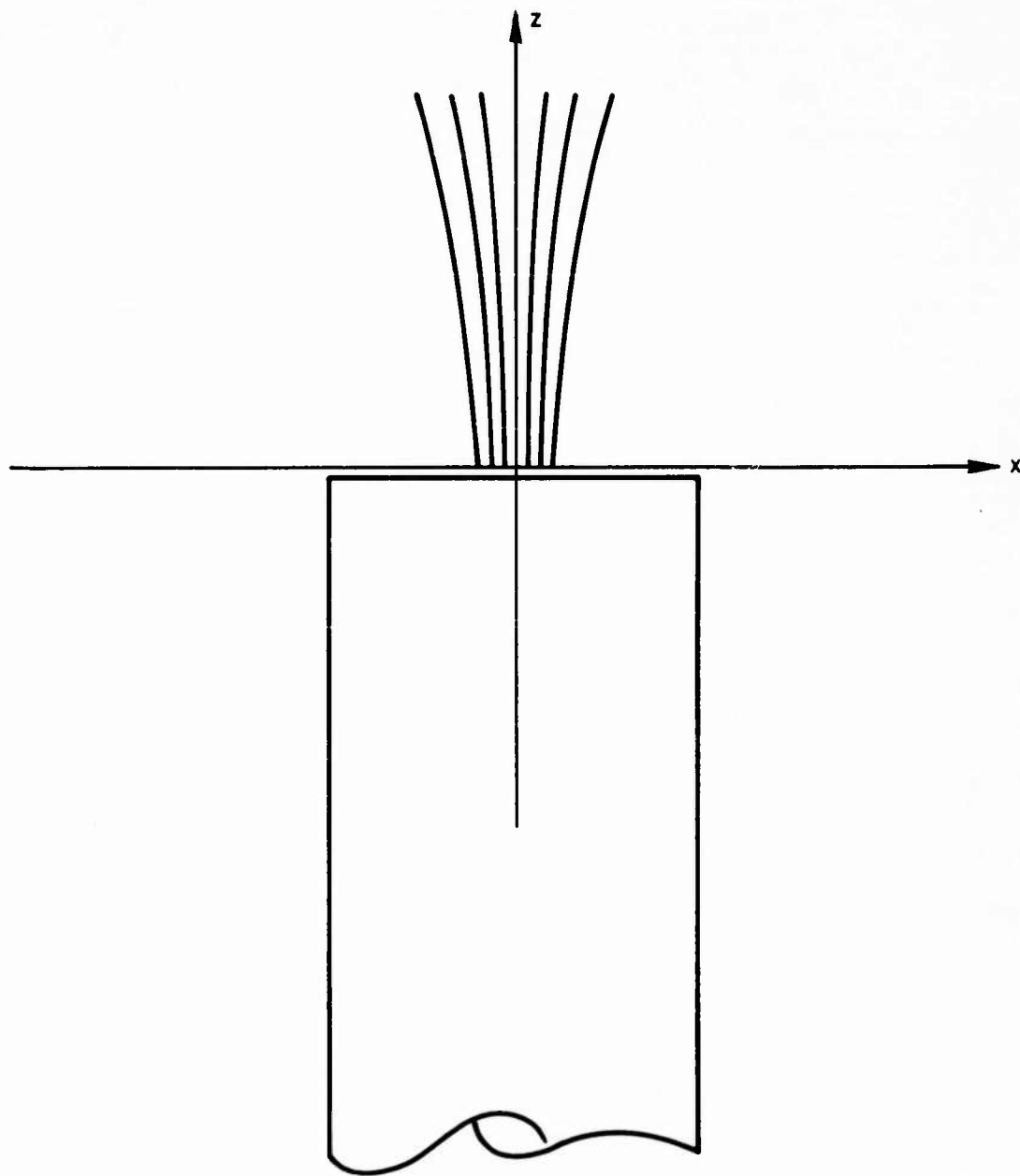
Fig. 39.   Nonuniform applied magnetic field geometry

REFERENCES

1.   M. Born and E. Wolf, Principles of Optics (Pergamon Press, Inc., New York, 1959), pp. 662-715.

2.   J. A. Stratton, Electromagnetic Theory (McGraw-Hill Book Co., Inc., New York, 1941).

3.   V. L. Ginzburg, Propagation of Electromagnetic Waves in Plasma (Gordon and Breach Science Publishers, Inc., New York, 1961).

4.   K. G. Budden, Radio Waves in the Ionosphere (Cambridge University Press, New York, 1961).

5.   J. A. Ratcliffe, The Magneto-Ionic Theory and Its Applications to the Ionosphere (Cambridge University Press, New York, 1959).

6.   H. G. Booker, "Some General Properties of the Formula of the Magneto-Ionic Theory," Proc. Roy. Soc. (London), A, 147 (1934), 352.

7.   I. P. Shkarofsky, "Generalized Appleton-Hartree Equation for Any Degree of Ionization and Application to the Ionosphere," Proc. Inst. Radio Engrs., (December, 1961),1857-1871.

8.   H. G. Booker, "Oblique Propagation of Electromagnetic Waves in A Slowly Varying Non-Isotropic Medium," Proc. Roy. Soc. (London), A, 155 (1936), 235.

9.   S. K. Mitra, "The Upper Ionosphere," Calcutta Asiatic Soc., 2nd Ed. (1952), 185-219.

10.   B. N. Gershman, V. L. Ginzburg, and N. G. Denisov, "Propagation of Electromagnetic Waves in Plasma (Ionosphere)," Uspekhi Fiz. Nauk, 61, (1957), 561, translated in AEC-TR-3493.

11.   M. L. V. Pitteway, "Reflection Levels and Coupling Regions in a Horizontally Stratified Ionosphere," Phil. Trans. Roy. Soc. (London), A, 252 (1959), 53.

12. S. K. Mitra, "Microwaves in the Ionosphere," Phys. Fluids, 3 (1960), 372.

13. V. Pisareva, "Method of Smooth Perturbations in Propagation Through a Medium with Inhomogeneities," Soviet Phys.-Acoustics, 81 (1960).

14. J. Haselgrove, "Ray Theory and a New Method for Ray Tracing," London Phys. Soc. Rept. of Conf. on the Physics of the Ionosphere (1954), 355.

15. J. Haselgrove, "Twisted Ray Paths in the Ionosphere," Proc. Phys. Soc. (London), 75, Pt. 3, No. 483 (1960).

16. H. O. Wüster and K. Försterling, "Über die Enstehung von Oberwellen in der Ionosphäre," J. Atmospheric and Terrest. Phys., 2 (1951), 22.

17. K. Försterling, "Über die Ausbreitung Elektromagnetischer Wellen in einem Magnetisierten Medium bei Senkrechter Incidenz," Hochfr. Elek., 59 (1942), 110.

18. P. C. Clemmow and J. Heading, "Coupled Forms of the Differential Equations Governing Radio Propagation in the Ionosphere," Proc. Cambridge Phil. Soc., 50 (1954), 319.

19. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis (Cambridge University Press, New York, 1952).

20. D. R. Hartree, "The Propagation of Electromagnetic Waves in a Stratified Medium," Proc. Cambridge Phil. Soc., 25 (1929), 97.

21. F. A. Albini and R. G. Jahn, "Reflection and Transmission of Electromagnetic Waves at Electron Gradients," J. Appl. Phys., 32, No. 1 (1961), 75-82.

22. G. F. Hermann, "The Absorption of Microwave Radiation in a Plasma Whose Electron Density Varies Linearly with Distance," General Telephone and Electronics Labs., Tech Rept. No. 2, RADC-TDR-62-87, 1962.

-114-

23. L. S. Taylor, "Microwave Reflection and Absorption by a Non-uniform Plasma Sheath," Proc. 2nd Symposium on Engineering Aspects of Magnetohydrodynamics (Columbia University Press, 1961), p. 360.

24. D. R. Hartree, "Optical and Equivalent Paths in a Stratified Medium Treated from a Wave Standpoint," Proc. Roy. Soc. (London), A, 131 (1931), 427.

25. M. V. Wilkes, "The Theory of Very Long Wireless Waves From the Ionosphere," Proc. Roy. Soc. (London), A, 175 (1940), 143.

26. O. E. H. Rydbeck, "On the Propagation of Radio Waves," Chalmers Tek. Högskol. Handl., No. 34 (1944).

27. O. E. H. Rydbeck, "The Reflection of Electromagnetic Waves From a Parabolic Ionized Layer," Phil. Mag., 34 (1943), 342.

28. W. Pfister, "Effect of the D-Ionospheric Layer on Very Low Frequency Radio Waves," J. Geophys. Research, 54 (1949), 315.

29. J. P. Stanley, "The Absorption of Long and Very Long Waves in the Ionosphere," J. Atmospheric and Terrest. Phys., 1 (1950), 65.

30. P. S. Epstein, "Reflection of Waves in an Inhomogeneous Absorbing Medium," Proc. Natl. Acad. Sci. U. S., 16 (1930), 627.

31. L. S. Taylor, "Reflection of a TE Wave From an Inverse Parabolic Ionization Density," IRE Trans. on Antennas and Propagation, AP-9, (1961), 582.

32. L. S. Taylor, "RF Reflectance of Plasma Sheaths," Proc. Inst. Radio Engrs., (December, 1961), 1831.

33. A. Vassiliadis, "Investigation and Study of Propagation in and Properties of Nonuniform Plasmas," Kane Engineering Labs, Second Quarterly Tech. Note, RADC-TN-61-238, 1961.

34. A. J. Penico, "Mathematical Methods in the Study of Wave Propagation in Inhomogeneous Media," RADC-TDR-62-86, 1 February 1962.

35.   G. Tyras and G. Heid, "On the Propagation of Electromagnetic Waves Through Anisotropic Layers," IRE Trans. on Antennas and Propagation, AP-7, Special Suppl. (December 1959), S296-S300.

36.   T. P. Harley and G. Tyras, "Transmission of Electromagnetic Waves Through an Ionized Layer in the Presence of a Strong Magnetic Field," Proc. Inst. Radio Engrs. (December, 1961), 1822-1824.

37.   G. Levy, "Propagation Properties of Nonuniform Ionized Media," RADC-TN-61-99, October, 1961.

38.   J. Richmond, "Transmission Through Inhomogeneous Plane Layers," Antenna Lab., Ohio State University, Rept. 1180-4, 15 July 1961.

39.   M. Klein, H. Greyber, J. King, and K. Brueckner, "Interaction of a Nonuniform Plasma with Microwave Radiation," GE MSVD R59SD467, November, 1959.

40.   J. J. Gibbons and R. J. Nertney, "A Method for Obtaining the Solutions of Ionospherically Reflected Long Waves," J. Geophys. Research, 56 (1951), 355.

41.   J. J. Gibbons and R. J. Nertney, "Wave Solutions, Including Coupling of Ionospherically Reflected Long Radio Waves for a Particular E-Region Model," J. Geophys. Research, 57 (1952), 323.

42.   K. G. Budden and P. C. Clemmow, "Coupled Form of the Differential Equations Governing Radio Propagation in the Ionosphere, II," Proc. Cambridge Phil. Soc., 53 (1957), 669.

43.   M. H. Cohen, "Magnetoionic Mode Coupling at High Frequencies," Astrophys. J., 131 (1960), 664.

44.   J. Heading, "The Reflection of Vertically Incident Long Radio Waves From the Ionosphere When the Earth's Magnetic Field Is Oblique," Proc. Roy. Soc. (London), A, 231 (1955), 414.

45.   J. Heading, "Analytical Consideration of Ionospheric Windows for Low Frequency Waves," J. Atmospheric and Terrest. Phys., 20, No. 1 (1961), 31-39.

46.   H. Poverlein, "Propagation of Electromagnetic Waves in a Plasma with a Strong Magnetic Field," Phys. Fluids, 4, No. 4 (1961), 397-405.

47.   M. V. Wilkes, "The Oblique Reflection of Very Long Wireless Waves From the Ionosphere," Proc. Roy. Soc. (London), A, 189 (1947), 130.

48.   J. Heading and R. T. P. Whipple, "The Oblique Reflection of Long Wireless Waves from the Ionosphere Where the Earth's Magnetic Field Is Regarded as Vertical," Phil. Trans. Roy. Soc. (London), A, 244 (1952), 469.

49.   M. P. Bachynski, I. P. Shkarofsky and T. W. Johnston, "Plasma Physics of Shock Fronts," RCA Victor Co., Ltd., Research Rept. No. 7-801.3, June, 1959.

50.   M. P. Bachynski, T. W. Johnston, and I. P. Shkarofsky, "Universal Representation of Electromagnetic Parameters in Presence of D. C. Magnetic Fields," Proc. Inst. Radio Engrs. (January, 1961).

51.   M. P. Bachynski, T. W. Johnston, and I. P. Shkarofsky, "Electromagnetic Properties of High Temperature Air," IRE Trans. on Antennas and Propagation, AP-7, Special Suppl. (December, 1959), S337-S339.

52.   R. Fante, "Propagation Through Plasmas in the Presence of a D.C. Magnetic Field," Avco Corp., EEDM-640-310, September, 1960.

53.   H. Hodara, "Use of Magnetic Fields in the Elimination of the Re-entry Radio Blackout," Proc. Inst. Radio Engrs. (December, 1961), 1825-1830.

54.   I. P. French, G. G. Cloutier, and M. P. Bachynski, "Electromagnetic Wave Propagation and Radiation Characteristics of Anisotropic Plasmas," RCA Victor Co., Ltd. Research Rept. No. 7-801.8, November, 1960.

55.   K. A. Graf and M. P. Bachynski, "Transmission and Reflection of Arbitrarily Polarized Electromagnetic Waves at the Boundary of an Anisotropic Plasma," RCA Victor Co., Ltd. Research Rept. No. 7-801.4, January, 1962.

56.   R. G. Jahn, "Interaction of Electromagnetic Waves with Ionized Gases," Calif. Inst. Tech. Rept. No. 9, August, 1959.

57.   R. G. Jahn, "Microwave Probing of Ionized Gas Flows," Calif. Inst. Tech. Note No. 5, February, 1962.

58.   R. Betchov and A. E. Fuhs, "A Survey of Diagnostic Methods for Ionized Gases Using Magnetic Fields," Aerospace Corp., Rept. TDR-169(3153)TR-1, September, 1962.

59.   J. Daiber and H. Glick, "Plasma Studies in a Shock Tube," CAL Rept. No. AF-1441-A-4, RADC TN 61-130, September, 1961.

60.   K. V. N. Rao, J. T. Verdeyen, and L. Goldstein,"Interaction of Microwaves in Gaseous Plasmas Immersed in Magnetic Fields," Proc. Inst. Radio Engrs. (December, 1961), 1877-1889.

61.   I. Jones and R. Gould, "An Experimental Study of the Propagation of Hydromagnetic Waves in a Plasma," Aerospace Corp. Rept. TDR-930(2210-02)TR-1, April, 1962.

62.   R. E. Lueg and J. E. Wood, "Microwave Attenuation in Ionized Gases Subject to a Magnetic Field," Univ. of Texas, E. E. Research Lab. Rept. No. 5-40, December 1959.

63.   H. Rothman and T. Morita, "Transmission Through an Ionized Medium in the Presence of a Strong Magnetic Field," AF Cambridge Research Labs., AFCRL 337, TR 74, May, 1961.

64. R. G. Delosh, "A Technique for Propagation Through a Plasma Sheath," Interim Rept. No. 3, Bendix Corp., BSR 667, 1 December 1961 - 1 March 1962.

65. E. L. Resler, S. C. Lin, and A. Kantrowitz, "The Production of High Temperature Gases in Shock Tubes," J. Appl. Phys., 23, No. 12 (1952), 1390-1399.

66. S. C. Lin, E. L. Resler, and A. Kantrowitz, "Electrical Conductivity of Highly Ionized Argon Produced by Shock Waves," J. Appl. Phys., 26, No. 1 (1955), 95.

67. L. Brekhovskikh, Waves in Layered Media (Academic Press, Inc., New York, 1960).

68. B. A. Spence, "Bibliography on Magnetohydrodynamics Plasma Physics and Controlled Thermonuclear Processes," Avco Everett, AMP 36, October, 1959.

69. G. R. Evans, "Electromagnetic Wave Propagation Through a Plasma; Theoretical and Experimental Studies," Lockheed Special Bibliography, SB-61-56, 3-80-61-34, January, 1962.

70. G. E. Owens, "Ionization and Radiation in Re-entry Flow Fields, A Selective Bibliography," Lockheed Special Bibliography, SB-61-34, 3-80-61-5, June, 1961.

71. C. O. Hines, "Heavy-Ion Effects in Audio Frequency Propagation," VLF Symposium Paper 11, Boulder, Colorado, 1957.

72. R. Gallet, "Microwave Whistler Mode Propagation in a Dense Laboratory Plasma," Phys. Rev. Letters, 7 (1960).

73. W. C. Hoffman, "Conditions for the Persistence of Purely Longitudinal or Purely Transverse Propagation," J. Atmospheric and Terrest. Phys., 18 (1960), 1-7.

UNCLASSIFIED

Aerospace Corporation, El Segundo, California.
ELECTROMAGNETIC WAVE PROPAGATION
THROUGH MAGNETOACTIVE PLASMAS, prepared
by R. Mason and R. R. Gold. 1 February 1962.
[122]p. incl. illus.
(Report TDR-69(2119)TR-3; DCAS-TDR-62-139)
(Contract AF 04(695)-69)   Unclassified report

The general problem of the propagation of a plane
harmonic electromagnetic wave through an ionized
gas is considered, including collisions, some form
of electron density distribution, and a magneto-
statically induced anisotropy. The present state of
the art is examined first, including an outline of
the WKB approximation and a summary of electron
density distributions for which complete wave
theory solutions exist. Then, specific analytical
solutions and appropriate numerical calculations
are derived to evaluate the effect of the applied
magnetic field on propagation. A detailed
(over)

UNCLASSIFIED

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

Aerospace Corporation, El Segundo, California.
ELECTROMAGNETIC WAVE PROPAGATION
THROUGH MAGNETOACTIVE PLASMAS, prepared
by R. Mason and R. R. Gold. 1 February 1962.
[122]p. incl. illus.
(Report TDR-69(2119)TR-3; DCAS-TDR-62-139)
(Contract AF 04(695)-69)   Unclassified report

The general problem of the propagation of a plane
harmonic electromagnetic wave through an ionized
gas is considered, including collisions, some form
of electron density distribution, and a magneto-
statically induced anisotropy. The present state of
the art is examined first, including an outline of
the WKB approximation and a summary of electron
density distributions for which complete wave
theory solutions exist. Then, specific analytical
solutions and appropriate numerical calculations
are derived to evaluate the effect of the applied
magnetic field on propagation. A detailed
(over)

Aerospace Corporation, El Segundo, California.
ELECTROMAGNETIC WAVE PROPAGATION
THROUGH MAGNETOACTIVE PLASMAS, prepared
by R. Mason and R. R. Gold. 1 February 1962.
[122]p. incl. illus.
(Report TDR-69(2119)TR-3; DCAS-TDR-62-139)
(Contract AF 04(695)-69)   Unclassified report

The general problem of the propagation of a plane
harmonic electromagnetic wave through an ionized
gas is considered, including collisions, some form
of electron density distribution, and a magneto-
statically induced anisotropy. The present state of
the art is examined first, including an outline of
the WKB approximation and a summary of electron
density distributions for which complete wave
theory solutions exist. Then, specific analytical
solutions and appropriate numerical calculations
are derived to evaluate the effect of the applied
magnetic field on propagation. A detailed
(over)

Aerospace Corporation, El Segundo, California.
ELECTROMAGNETIC WAVE PROPAGATION
THROUGH MAGNETOACTIVE PLASMAS, prepared
by R. Mason and R. R. Gold. 1 February 1962.
[122]p. incl. illus.
(Report TDR-69(2119)TR-3; DCAS-TDR-62-139)
(Contract AF 04(695)-69)   Unclassified report

The general problem of the propagation of a plane
harmonic electromagnetic wave through an ionized
gas is considered, including collisions, some form
of electron density distribution, and a magneto-
statically induced anisotropy. The present state of
the art is examined first, including an outline of
the WKB approximation and a summary of electron
density distributions for which complete wave
theory solutions exist. Then, specific analytical
solutions and appropriate numerical calculations
are derived to evaluate the effect of the applied
magnetic field on propagation. A detailed
(over)

Aerospace Corporation, El Segundo, California.
ELECTROMAGNETIC WAVE PROPAGATION
THROUGH MAGNETOACTIVE PLASMAS, prepared
by R. Mason and R. R. Gold. 1 February 1962.
[122]p. incl. illus.
(Report TDR-69(2119)TR-3; DCAS-TDR-62-139)
(Contract AF 04(695)-69)   Unclassified report

The general problem of the propagation of a plane
harmonic electromagnetic wave through an ionized
gas is considered, including collisions, some form
of electron density distribution, and a magneto-
statically induced anisotropy. The present state of
the art is examined first, including an outline of
the WKB approximation and a summary of electron
density distributions for which complete wave
theory solutions exist. Then, specific analytical
solutions and appropriate numerical calculations
are derived to evaluate the effect of the applied
magnetic field on propagation. A detailed
(over)

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.

parametric investigation, several exemplary flight conditions, and a possible experimental shock-tube program are analyzed assuming a homogeneous plasma. Preliminary estimates of the effects of plasma inhomogeneities and nonuniformities in the magnetic field on propagation are obtained.